ЗАЩИТА ОТ ЭЛЕКТРОМАГНИТНЫХ ИЗЛУЧЕНИЙ

AN OBJECTIVE IN EDUCATION*

C. C. MacDUFFEE, University of Wisconsin

1. A philosophy. Early in February a mimeographed letter was sent to more than a hundred national and sectional officers of the Mathematical Association of America asking for suggestions in formulating the policy of the Association with respect to post-war educational problems. The first and primary question was, "Do you believe that the Association should, through the action of its committees, keep in close touch with developments in the educational world and take an active part in moulding opinion?" The reply to this question was an emphatic and unanimous "yes." Since those replying undoubtedly constitute a fair sample of the total membership of the Association, the officers feel that they have a mandate for action.

In these days when education seems to have so many aims and objectives, it was interesting to note that the members of the Association are pretty thoroughly agreed upon a fundamental philosophy of education. This is not, as one might perhaps have believed, that all students be required to take mathematics. It is, on the contrary, the belief that education should be tailored to the capacity of the individual. In this belief the members of the Association seem to have the support of many educators in other fields.

While this talk derived much inspiration from the replies to the above mentioned letter, I wish to assume full responsibility for the remarks that follow. The Association is in no degree committed to agree with them.

2. A problem. The philosophy of an education fitted to the capacity of the student has, up to the present, been largely developed in the interests of the weaker students. This is of course an essential step in the unique experiment in universal education for which the schools of the United States are now a vast laboratory. It is an experiment which would be attempted only by a great democracy, and it is an experiment which must be successfully completed if democracies are to survive and flourish.

But in solving the problem of the poorer student, difficulties have been placed in the path of the superior student which in some instances make it difficult for him to obtain the education of which he is capable. Because the superior students are in the minority, and because they do not become problem children, they are often woefully neglected. Because advanced subjects in a small high school have small registration, they are often abandoned and the time of the teacher is "more economically employed" with large classes of "future citizens."

It seems evident that the people of the United States do not see with sufficient clarity that the education of leaders is an objective compared with which the education of the masses is of lesser importance. This may sound like heresy in modern America, but it is an historical fact. Nations have become prominent

* An address delivered before a joint meeting of the Wisconsin Section of the Mathematical Association of America, the Mathematics Section of the Wisconsin Education Association and the Mathematics Club of Milwaukee at the Milwaukee State Teachers College on May 5, 1945.

and prosperous without universal education, but no nation has ever been able to maintain its position as a civilized community without a group of talented and educated leaders. America can and must solve both problems.

Let us recall that the history of the world is largely the history of individual men, many of whom are remembered only through their works. Modern civilization has been developed and is being maintained by relatively few persons. For each branch of science, the arts, philosophy, government, and military science it would be possible to name a hundred men without whose contributions the world would be appreciably poorer. Where would American technology now be if every person listed in the biographical book *American Men of Science* had failed to attend college? It is interesting to speculate on the present state of our civilization if one man in a thousand, since 10,000 B.C., properly selected, had failed to mature intellectually. It is quite possible that we should still be roaming the woods in skin clothing. Whether we would be happier in that condition than we now are is beside the point.

It is pertinent to note that the present apex of American military might was achieved without compulsory peace-time military training. A vitally important factor in our success was the work of American scientists in developing new weapons of offense and defense. Indeed, this was probably the decisive factor in the war—a fact which will some day be clearer than it is now to the general public. If compulsory military training has the effect of delaying or in any way interfering with the development of our skilled scientists, the act will be as disastrous as the destruction of the proverbial goose with the talent for laying golden eggs.

Even if we suppose that the gifted student eventually reaches the university, the years of time which he has lost are never recovered. In no direction is American wastefulness more costly than in the waste of precious years in the schooling of the superior child. Contrary to popular superstition, it does not injure the brain of a child to allow him to absorb knowledge at his natural rate of speed, even though this be five times the rate of his slightly retarded classmates. On the other hand, to allow a superior child to coast along without the exercise of effort is distinctly detrimental.

The fact that Europe has produced men of great genius more abundantly than has America cannot be due to differences in native ability, for we are of the same stock. It is clearly due to the fact that the better European schools are geared to the pace of students of higher average ability, and about two years are saved in their elementary education. These two precious years which American students lose are never recovered, and are subtracted from their years of greatest productivity.

There are some who, with a befuddled conception of the meaning of democracy, argue that a special education for the gifted means the production of an intellectual aristocracy and is therefore undemocratic. The effect of such thinking is to limit all opportunities of leadership to those men whose families are sufficiently wealthy to send their sons to private schools and colleges, a plan

which is wasteful of precious talent and is the very antithesis of democracy. Genius does frequently appear among the children of the poor, and a truly democratic educational system will provide adequate education for the development of such genius.

In advocating the double-track curriculum in the secondary schools, we must be careful not to call the express track the "college preparatory" track. A superior student in high school has a right to a superior education even if he is financially unable to attend college. The fact that his financial situation is more subject to alteration than is his mental equipment should impel a teacher to be very cautious about advising a capable student to elect a vocational course. Many of life's little tragedies occur when students discover that their high school work is of no use to them in preparation for their college work.

3. A solution. It is far easier to state a problem than to solve it, and I have to approach this problem cautiously. The City of New York has reached a satisfactory solution with its highly trained and adequately paid teachers, its special schools for the gifted as well as for the handicapped, and its system of city colleges which admit only those graduates of the local high schools who have attained high grades in a substantial college-preparatory course. But the small high schools of northern Wisconsin, for instance, present a problem which cannot be solved in the same manner.

One possibly satisfactory answer to this problem would be in the establishment of a few centralized schools throughout the state which would take the better student at the tenth grade level and fit him for college. This idea is not new; it is, to take one instance, the motivation of the establishment of the college at the University of Chicago. President Hutchins sees very clearly that the last two years of high school are frequently very wasteful of a talented student's precious years of youth, and he has provided an answer.

A means of accomplishing this end might be the junior college offering four years of work from the eleventh to the fourteenth grades inclusive. The junior college movement had considerable impetus a few years ago and this has not been entirely lost. But if the junior college is merely to offer a diluted form of college education for the student who is too weak for the university, it will not serve the purpose with which we are now concerned. Such an institution would merely extend the student's high school education by two years. If the junior college is to be the solution of the problem of the superior student, it must have high entrance requirements, a substantial curriculum from which the unessential courses have been eliminated, and a competent faculty who are able to stimulate and enthuse the students.

At the beginning of the eleventh grade the tone of the school should change from the juvenile to the adult. The student by this time should be made to realize that scholarship is a dignified career, fully as worthy of masculine attention as football or class politics. That the scholars are the people who keep the sacred fire of culture burning is a fact which merits as much time for indoctrina-

tion as the nutritional value of cheese.* The age in which we live is a serious age. Competition between men and nations is deadly. The transition which our eighteen-year-olds had to make from the childish atmosphere of the school to the brutality of army life might have been easier if the schools had not for years pretended that realities need not be faced, that difficult subjects may be skipped, that students who fail must nevertheless be passed along for fear of consequences to the personality. The epoch of such nonsense is past, and I believe that the American people are now in a mood to face facts, and to demand that their children be prepared in the schools to face life as it is, not a fairy-land life in which one can avoid the consequences of stupidity and laziness.

4. A liberal education. For many years a college education meant the traditional course in liberal arts with Latin and Greek and the Roman and Hellenic cultures as a background. It was and still is a most excellent course, demanding that the student give his best efforts and rewarding him with a stereoscopic view of our language, history and civilization. The universities must preserve this course for those who will take it.

In the last two hundred years, our civilization has completely changed. Sciences which were unknown in 1700 now determine the fate of nations and guide the daily lives of men. While the traditional liberal arts course is still important, it is not sufficient as it once was to interpret to the individual the world in which he lives. If the universities would reexamine and restate their objectives, perhaps the high schools would be able to see more clearly what their procedure should be.

I believe that from now on the central course of study in the colleges should be a course in liberal science not leading to any specialization but serving to interpret to the student the marvelous civilization of which we all are conscious, but which is incomprehensible at least in part to most of us. I can see the raising of eyebrows by my colleagues in the humanities, the familiar charge that the mere scientist is a technician, not a truly educated man who feels the finer things of life. I am not impressed. Unless the humanitarian understands the forces which keep the earth in its orbit, he is not a truly educated man. The pure sciences, taught with the objective of explaining basic principles, have as much aesthetic value as the humanities, and they have a grandeur all their own.

This course in liberal science should include work in physics and chemistry, and at least an introduction to astronomy, geology, biology and physiology. These should not be of the I-can't-tell-you-why-because-you-wouldn't-understand-it type of course, but courses where facts are proved. Astronomy, that now neglected gem of the sciences, should be highly recommended. There is no course which the humanitarians can offer which has quite the same power to orient a student with respect to his environment and reduce him to a position of humility as astronomy. Mathematics through differential equations is, of course, essential for all the physical sciences.

* There is an uninforced law in Wisconsin that all pupils shall be instructed each week in the salubrity of dairy products.

It goes without saying that the course should include English and a modern language and history, but possibly all of these courses can be given the emphasis demanded by their position in a course in liberal science. Composition and speech remain important, but the detailed study of English literature may have to be sacrificed as was the literature of classic times. Similarly language study should give more attention to the scientific literature than to archaic forms. History should include a description of modern research in pre-history and much emphasis on the history of science. In fact, a course in the senior year in the history of science given to students who are familiar with the sciences should be one of the high spots which make the course a success. Have you a pet course which should be included? There is room for electives.

An engineering course of two years' duration following a course in liberal science such as I have described would produce a man well-rounded and generally educated, and technologically superior. The investment of an extra two years of study would be well worth while and as a matter of fact the double-track scheme of elementary and high school education could save for the superior child the two extra years required.

The vogue of education-by-stuffing is definitely on the wane. The great corporations who now take so many of the graduates of our engineering schools have come to desire the students of superior ability who have done well in the basic science courses. Courses in engineering practice are of some importance, but many corporations prefer to initiate the junior engineer into their own methods and techniques. Mathematics and the basic sciences are the subjects which count. All this argues in favor of the course in liberal science.

5. Mathematics. I am not going to put forth any arguments in favor of the teaching of mathematics in our public schools. If there is anyone present who does not realize that we are living in the age of science, and that science rests on a mathematical foundation, I know of no language by which I can reach him. There are in our schools students who cannot learn mathematics. It is doubtful if they are capable of fundamental thinking in any direction. They are destined to be drawers of water and hewers of wood in our civilization. They should be given generous and sympathetic attention so that they may become useful citizens. They will not become leaders.

There are other students who have in some way acquired a distaste for mathematics and a conviction that they are unable to master it. They are the casualties of poor teaching. Under normal conditions they pass through life firm in the conviction that they are not mathematically minded. During the days of the ASTP I encountered numerous instances which showed exactly where the trouble lay. These boys have many times assured me that they hated mathematics, had no ability for it, and that only the power of the Army of the United States could persuade them to take it. But in spite of the fact that the ASTP curriculum was an educational monstrosity, many of these boys became mathematical fans. Recently I received two letters from the Pacific theatre from boys who had been in my ASTP class in calculus. One stated that he wanted to return

to Wisconsin and take all the courses in mathematics and chemistry that were given. The other asked for the title of a book on non-euclidean geometry.

If I were asked for suggestions for the improvement of the teaching of mathematics in high school, I should say that we need more teachers who have a burning enthusiasm for the subject. If they love the subject for its own sake, they will continue to study it during their teaching years, and they will be able to communicate their enthusiasm to the student. Certainly mathematics is important and useful, but more than that, it is fun. This is the motive which, more often than any other, determines a person's choice of vocation. Many majors in the university are determined by the personality of a favorite teacher.

Those of us who appreciate mathematics should insist that more mathematics, not less, be offered in the post-war schools and that all students who can profit by it be encouraged to partake. There is strong evidence that the war has awakened America from its world of fancy and that the country is ready to face realities. Mathematics has been belittled by some educational experimenters in the last twenty years, but the demands of the war have brought confusion upon them. As Kipling stated it,

"It's 'Tommy' this, and 'Tommy' that,
And 'Throw him out, the brute!'
But it's 'Savior of his country'
When the guns begin to shoot."

6. Looking ahead. The four years of mathematics which high schools that wish to be designated as college-preparatory must offer should prepare a student to begin calculus upon entering college. In no other way can the student who takes physics in his freshman year avoid the loss of a year. Physics taught without calculus is a lean and ineffectual thing, and I sometimes believe that it does more harm than good in fixing rationalizations rather than fundamental methods in the student's mind. The synchronization of courses in physics and mathematics is one of the problems now crying for solution. This was done to some extent in the Army and Navy programs and was, I believe, one of the high spots in the programs.

Analytics is now being taught in the fourth year in some of our large city high schools, in many preparatory schools, and in the schools of Ontario. It is not an untried experiment. It supplants the traditional course in advanced algebra, accomplishes the same objective as regards manipulative skill, and by furnishing a motive, increases the interest of the student.

There is one objection which is brought up against the teaching of analytic geometry in high school, and that is, as one man expressed it, "The first persons to study it would have to be the teachers." If that is true, I think I am in favor of putting analytic geometry into all the high schools, for I am sure that all high school teachers of mathematics should know analytics. If teachers were recruited from the ranks of those who had taken the course in liberal science which I described, they would be thoroughly qualified in technical skill, breadth of vision and, I believe, enthusiasm.

7. Summary. It appears that more and more educators are abandoning the thesis that all students should be forced to go as far as they can through the college preparatory high school courses. The principle of the course tailored to fit the capacity of the student seems to be attracting more and more adherents, not only among high school teachers, but also among college teachers. This plan of a double-track curriculum promises excellent results provided the express track is not neglected in favor of the freight track, and provided that students of ability are persuaded to purchase Pullman tickets and not bills of lading.

There can be no real permanent threat to the teaching of mathematics in our schools, since our culture is essentially a scientific culture resting upon a mathematical base. The best way that mathematics teachers can circumvent ill-considered attacks upon mathematics is to support sound educational principles in general without too much emphasis upon mathematics in particular. If collegiate instruction were built around a non-specialized course designed to interpret modern civilization to the student, such a course would necessarily contain courses in science and mathematics. It would require in preparation a full four years of mathematics in high school, and would provide an ideal training for prospective high school teachers of science and mathematics.

Irrespective of how our problems are eventually to be solved, it seems desirable that teachers of mathematics at all levels should now work together in the interests of sound education, and should join forces with persons of similar ideals and objectives in other branches of learning. United behind well-defined objectives, the educational societies of America will be able to keep education in this country on a rational basis.

CONVERGENCE OF MULTIPLY-INFINITE SERIES

I. M. SHEFFER, Pennsylvania State College

1. Introduction. There are various definitions of convergence for multiply-infinite series. Generally speaking, a definition first defines a class P of permissible partial sums (a partial sum always consisting of a finite number of terms). Then the series is said to converge to the sum A if to every $\epsilon>0$ there corresponds a suitable subclass P_ϵ of partial sums taken from P, such that the inequality

$$|A - s| < \epsilon \tag{1.1}$$

holds for every partial sum s belonging to P_ϵ.

The class of series that converge according to a given definition will vary with the class P of permissible partial sums, and as class P is more and more restricted, the class of corresponding convergent series will be more and more inclusive.

An example of a widely used definition (the *Pringsheim* definition) is that

where P consists of all *rectangular sums*. Let the series be $\sum a_{ij}$, where double series are used for simplicity. By R_{pq} is meant the rectangular sum

$$R_{pq} = \sum a_{ij} \qquad (0 \leqq i \leqq p; 0 \leqq j \leqq q); \tag{1.2}$$

and $\sum a_{ij}$ is said to converge to A if to every $\epsilon > 0$ there correspond indices p, q such that

$$|A - R_{rs}| < \epsilon \tag{1.3}$$

for all $r > p$, $s > q$.

If a series converges according to this definition, the simple series forming the terms of a given row or column need not converge. For example, the definition gives convergence (to the sum 0) to the series whose first two rows are

$$0! + 1! + 2! + \cdots + n! + \cdots$$
$$- (0!) - (1!) - (2!) - \cdots - (n!) - \cdots$$

and where all other terms are zero. Each of the first two rows is highly divergent, which is undesirable for some purposes.

At the other extreme is the following definition: Series $\sum a_{ij}$ converges if no matter how the series is arranged to form a simple series, the resulting simple series converges, and to a sum that is independent of the order in which the terms are taken. This definition permits P to be the class of *all* possible partial sums, and the condition for convergence can be phrased as follows: Series $\sum a_{ij}$ converges to A if to every $\epsilon > 0$ correspond indices p, q such that $|A - s| < \epsilon$ for every partial sum s that includes R_{pq}. This definition is highly restrictive; it is, in fact, a definition that implies absolute convergence.

It is our purpose to give a definition of convergence that lies between the two above-mentioned definitions, one that possesses properties analogous to those of simple series; and to develop some of these properties. Where proofs are especially direct and simple we shall often omit them. For simplicity in notation, the definition of convergence, and the theorems and proofs that follow, will be stated for the case of double series. The extension to the general case of k-tuply infinite series is usually immediate; where this is not the case, further details will be found in §III.

2. Definition and properties of σ-sum convergence. Let the series $\sum a_{ij}$ be given, where i, j run independently from 0 to infinity.

DEFINITION. *A partial sum is said to be a sigma-sum (σ-sum) if it has the following property: If it contains a_{pq} then it also contains the rectangular sum*

$$R_{pq} = \sum a_{ij} \qquad (0 \leqq i \leqq p, 0 \leqq j \leqq q). \tag{2.1}$$

The reader can exhibit the nature of a σ-sum (for double series) as follows: Let all the squares occupied by the terms of a σ-sum be shaded. Then the boundary of the shaded region is a "staircase," in general irregular, *that rises to the right.*

We take the class P to be the class of all σ-sums.

DEFINITION. *Given indices p, q, a σ-sum is said to be a sigma-sum relative to (p, q) [written σ-sum(p, q)] if it contains a_{pq}.*

COROLLARY. *A σ-sum(p, q) contains R_{pq}.*

DEFINITION. *Series $\sum a_{ij}$ converges to the sum A if to every $\epsilon>0$ there correspond indices (p, q) such that*

$$|A - \sigma| < \epsilon \tag{2.2}$$

for every σ that is a σ-sum(p, q).

COROLLARY. *If $\sum a_{ij}$ converges, it has a unique sum.*

Since rectangular sums are also σ-sums, we conclude that convergence by rectangular sums is at least as inclusive as is the present definition. Accordingly we have

LEMMA 1. *If a series converges by σ-sums, then it converges also by rectangular sums, and to the same sum.*

We shall see that the converse does not hold.

LEMMA 2. *If $\sum a_{ij}$ converges, so does $\sum \lambda a_{ij}$; and*

$$\sum \lambda a_{ij} = \lambda \sum a_{ij}. \tag{2.3}$$

THEOREM 1 (*Cauchy criterion*). *A necessary and sufficient condition that series $\sum a_{ij}$ converges is that to every $\epsilon>0$ there correspond indices (p, q) such that*

$$|\sigma' - \sigma''| < \epsilon \tag{2.4}$$

for every pair σ', σ'' of σ-sums(p, q).

First suppose that the series converges, to the sum A. Then given $\epsilon>0$, indices (p, q) exist so that $|A-\sigma'|<\epsilon/2$, $|A-\sigma''|<\epsilon/2$ whenever σ', σ'' are σ-sums(p, q). Hence (2.4) holds. Now suppose (2.4) holds; to show that the series converges. If σ' is fixed, (2.4) shows that the set $\{\sigma''\}$ of all σ-sums(p, q) is bounded, and therefore has at least one limit point A. Accordingly, there is a σ-sum(p, q), say σ''', such that $|A-\sigma'''|<\epsilon$; and from (2.4), $|A-\sigma|<2\epsilon$ for every σ-sum(p, q). From the arbitrariness of ϵ we conclude that $\sum a_{ij}$ converges to A.

THEOREM 2. *If $\sum a_{ij}$ converges, then to every $\epsilon>0$ correspond indices (p, q) such that*

$$|a_{ij}| < \epsilon \tag{2.5}$$

for every a_{ij} not in R_{pq}.

Let (p, q) be chosen so that $|A-\sigma| < \epsilon/2$ for every σ-sum(p, q), A being the sum of the series. If a_{ij} is not in R_{pq}, it can be written as the difference of two σ-sums(p, q):

$$a_{ij} = \sigma' - \sigma''.$$

In fact, σ' can be chosen as R_{ij} augmented by all the terms of R_{pq} not already in R_{ij}; and σ'' is then obtained from σ' by depriving it of the term a_{ij}. That σ', σ'' as so chosen are σ-sums(p, q) is readily ascertained. Then,

$$|a_{ij}| = |\sigma' - \sigma''| \leqq |A - \sigma'| + |A - \sigma''| < \epsilon,$$

which is (2.5).

COROLLARY. *If* $\sum a_{ij}$ *converges, then*

$$\lim a_{ij} = 0 \qquad (i + j \to \infty). \tag{2.6}$$

From the example of §1 we conclude the further

COROLLARY. *Rectangular sum convergence does not imply* σ*-sum convergence.*

THEOREM 3. *If* $\sum a_{ij}$, $\sum b_{ij}$ *converge, then also* $\sum(a_{ij}+b_{ij})$ *converges, and*

$$\sum a_{ij} + \sum b_{ij} = \sum (a_{ij} + b_{ij}). \tag{2.7}$$

THEOREM 4 (*Comparison theorem*). *If* $0 \leqq a_{ij} \leqq b_{ij}$, *and if* $\sum b_{ij}$ *converges, so does* $\sum a_{ij}$, *and* $\sum a_{ij} \leqq \sum b_{ij}$.

For let $B = \sum b_{ij}$. To $\epsilon > 0$ correspond indices (p, q) so that $|B-\sigma'| < \epsilon$ where σ' is any σ-sum(p, q). If σ is the corresponding sum in $\sum a_{ij}$, then $\sigma \leqq \sigma'$. Since, as is easily seen, $\sigma' \leqq B$, therefore $\sigma \leqq B$. The set $\{\sigma\}$ is consequently bounded, and has a least upper bound A. We shall show that $\sum a_{ij}$ converges to A.

A σ_1 exists so that $0 \leqq A - \sigma_1 < \epsilon$. If indices (r, s) are then chosen so that σ_1 is contained in R_{rs}, it follows that for every σ that is a σ-sum(r, s), we have $\sigma_1 \leqq \sigma$ and therefore $0 \leqq A - \sigma < \epsilon$. Hence $\sum a_{ij}$ does converge to A; and, of course, $A \leqq B$.

COROLLARY. *If* $0 \leqq b_{ij} \leqq a_{ij}$, *and if* $\sum b_{ij}$ *diverges, then also* $\sum a_{ij}$ *diverges.*

THEOREM 5. *Let* $a_{ij} = a'_{ij} + \sqrt{-1}\, a''_{ij}$, *where* a'_{ij} *and* a''_{ij} *are real. Then* $\sum a_{ij}$ *converges if and only if* $\sum a'_{ij}$, $\sum a''_{ij}$ *converge; and in this case,*

$$\sum a_{ij} = \sum a'_{ij} + \sqrt{-1} \sum a''_{ij}. \tag{2.8}$$

DEFINITION. *Series* $\sum a_{ij}$ *is said to converge absolutely if* $\sum |a_{ij}|$ *converges.*

THEOREM 6. *If* $\sum a_{ij}$ *converges absolutely, then it converges.*

Case 1. a_{ij} *real.* Let $\sum a'_{ij}$ be the series obtained from $\sum a_{ij}$ by replacing all negative terms by 0, and $\sum a''_{ij}$ by replacing all non-negative terms by 0 and changing the sign of the remaining terms. By Theorem 4, $\sum a'_{ij}$, $\sum a''_{ij}$ both converge; and by Lemma 2 and Theorem 3, so does $\sum(a'_{ij} - a''_{ij})$, which is to say, $\sum a_{ij}$.

Case 2. a_{ij} complex. Let $a_{ij}=a'_{ij}+\sqrt{-1}a''_{ij}$. Then $|a'_{ij}| \leqq |a_{ij}|$, $|a''_{ij}| \leqq |a_{ij}|$, so that $\sum a'_{ij}$, $\sum a''_{ij}$ both converge absolutely. By Case 1, they also converge, whence by Theorem 5, $\sum a_{ij}$ converges.

LEMMA 3. *If $\sum a_{ij}$ converges absolutely either by the rectangular sum definition or by the σ-sum definition, it converges absolutely by the other.*

Lemma 1 establishes the assertion in the direction of σ-sum to rectangular sum. Now suppose that $\sum |a_{ij}|$ converges to A^* by rectangular sums. To $\epsilon>0$ corresponds an R_{pq} so that $0 \leqq A^*-R_{pq}<\epsilon$; hence for every σ-sum(p, q) for series $\sum |a_{ij}|$, $0 \leqq A^*-\sigma<\epsilon$. That is, $\sum |a_{ij}|$ converges by the σ-sum definition.

LEMMA 4. *Convergence does not imply absolute convergence.*

This is shown by the following double series. Let the first row be the series

$$1-\frac{1}{2}+\frac{1}{3}-\cdots+\frac{(-1)^n}{n+1}+\cdots$$

while all other terms are zero. As a double series this converges to the same sum as the simple series in the first row; but it is not absolutely convergent.

THEOREM 7. *Let $\sum a_{ij}$ converge to the sum A. If i (if j) is fixed, the resulting row series (column series) converges; and if its sum is denoted by A_{i*} (by A_{*j}), then also the series $\sum A_{i*}$ (series $\sum A_{*j}$) converges, and its sum is A.*

In other words, all rows and all columns converge, and the series can be summed by summing either the rows or the columns. To prove this we shall work with the rows for definiteness.

Let $i=0$; that is, consider the first row. To each $\epsilon>0$ there correspond indices (p, q) such that $|\sigma_1-\sigma_2|<\epsilon$ for every pair of σ-sums(p, q). Choose σ_1 as the rectangular sum R_{pq}, and σ_2 as R_{pq} augmented by the sum $\sum_{q+1}^{n} a_{0j}$; then $|\sum_{q+1}^{n} a_{0j}|<\epsilon$. Since n may be chosen as any integer exceeding q, it follows that $|\sum_{m}^{n} a_{0j}|<2\epsilon$ for all m, n for which $q<m<n$; and this implies the convergence of $\sum a_{0j}$.

Now let $i=1$. Let σ_1 again be R_{pq}, and now let σ_2 be R_{pq} to which have been added the terms $\sum_{q+1}^{n}(a_{0j}+a_{1j})$. Then the last sum is in magnitude less than ϵ. Since $|\sum_{q+1}^{n} a_{0j}|<\epsilon$, as already shown, therefore $|\sum_{q+1}^{n} a_{1j}|<2\epsilon$, and $|\sum_{m}^{n} a_{1j}|<4\epsilon$; that is, $\sum a_{1j}$ converges. And so on for every row: all rows converge.

There remains to show that $\sum A_{i*}$ converges, and to the sum A. Let $\epsilon>0$ be given. Then indices (r, s) exist such that

$$\text{(a)} \qquad |A-\sigma|<\frac{\epsilon}{2}$$

for all σ-sums(r, s). Let p be an integer at least as great as r. Since all rows converge, there is an index $q>s$ (q depending on p) such that

(b) $$\left| A_{m*} - \sum_{j=0}^{t} a_{mj} \right| < \frac{\epsilon}{2(p+1)} \qquad \begin{pmatrix} m = 0, 1, \cdots, p; \\ t = q, q+1, q+2, \cdots \end{pmatrix},$$

hence

(c) $$\left| \sum_{0}^{p} A_{m*} - \sigma \right| < \frac{\epsilon}{2},$$

where $\sigma = \sum_{m=0}^{p} \sum_{j=0}^{t} a_{mj}$ is a σ-sum(r, s). In conjunction with (a), this leads to

(d) $$\left| A - \sum_{0}^{p} A_{m*} \right| < \epsilon.$$

As p is arbitrary (subject to the condition $p \geqq r$), we conclude that $\sum A_{m*}$ converges to the sum A.

In §1 an example of a double series was given which converges by the rectangular sum definition, but for which a row is divergent. Hence Theorem 7 is not true for rectangular sum convergence.

Consider the double series

$$\begin{aligned} &1 - 1 + 0 + 0 + \cdots \\ &+ 0 + 1 - 1 + 0 + 0 + \cdots \\ &+ 0 + 0 + 1 - 1 + 0 + 0 + \cdots \\ &\cdots\cdots\cdots\cdots\cdots \end{aligned}$$

where in each row the only two non-zero terms are 1, -1. Each row converges to the sum 0, so $A_{i*} = 0$, $\sum A_{i*} = 0$. The first column sums to 1, and the others to 0, giving to $\sum A_{*j}$ the sum 1. Therefore by Theorem 7 the series does not converge. This result may be explained by the fact that not all series involved are absolutely convergent. This is illustrated by

THEOREM 8. *Given series $\sum a_{ij}$. If for each fixed i the simple series $A_{i*} = \sum_j |a_{ij}|$ converges, and if $\sum A_{i*}$ converges, then the original series $\sum a_{ij}$ converges absolutely; and correspondingly for columns.*

It suffices to consider the double series $\sum |a_{ij}|$. Let σ be any σ-sum (for $\sum |a_{ij}|$). Clearly, $\sigma \leqq \sum A_{i*}$, so that the set of all σ-sums is bounded, and therefore has a least upper bound, say L. Let $\epsilon > 0$ be given. There is a σ-sum, call it σ_1, such that $0 \leqq L - \sigma_1 < \epsilon$. Choose (p, q) so that σ_1 is contained in R_{pq}. Then for every σ-sum(p, q), $0 \leqq L - \sigma < \epsilon$. In other words, $\sum |a_{ij}|$ converges to the sum L.

DEFINITION. *If the first p rows and q columns are deleted from the series $\sum a_{ij}$, the new series, which has a_{pq} as its leading term, will be termed the truncate of order (p, q) relative to the original series.*

THEOREM 9. *If series $\sum a_{ij}$ converges, so does every truncate series.*

The truncate (0, 0) converges since it is the original series. Consider, then, the truncate (p, q), where either $p \neq 0$ or $q \neq 0$; and for definiteness suppose $p > 0$. It is a straightforward matter, that we leave to the reader, to show that truncate (1, 0) converges. It then follows by applying the same argument to series (1, 0), that (2, 0) converges; and so on until the convergence of $(p, 0)$ is obtained. Now turning to the second index, the same argument, repeated, shows that the series $(p, 1)$, $(p, 2)$, $\cdots$, and finally (p, q), converge.

As is to be anticipated, the property of Theorem 9 is false for rectangular sum convergence. The double series example of §1 shows this, since the truncate (1, 0) is clearly divergent.

THEOREM 10. *Let the double power series* $\sum a_{ij}u_1^i u_2^j$ *converge for* $u_s = \xi_s$, $s = 1, 2$. *Then it converges absolutely for every* (u_1, u_2) *for which* $|u_s| < |\xi_s|$, $s = 1, 2$, *and converges uniformly in every closed region therein, thus representing an analytic function of the variables* u_1, u_2 *in* $|u_s| < |\xi_s|$, $s = 1, 2$.

The theorem is vacuously true if $\xi_s = 0$ for at least one s. We therefore assume that $|\xi_s| > 0$, $s = 1, 2$. Choose r_s, $s = 1, 2$ so that $0 < r_s < |\xi_s|$, and restrict u_1, u_2 to the region $|u_s| \leqq r_s$. By Theorem 2, an M exists so that $|a_{ij}\xi_1^i \xi_2^j| \leqq M$. Hence

$$|a_{ij}u_1^i u_2^j| \leqq M\left(\frac{r_1}{|\xi_1|}\right)^i \cdot \left(\frac{r_2}{|\xi_2|}\right)^j.$$

The right-hand member is the general term of a convergent series. In fact if we sum on one index at a time (Theorem 8), we arrive at the sum $M \cdot \Pi_{s=1}^{2}\{1 - r_s/|\xi_s|\}^{-1}$. The desired result now follows from Theorem 4.

COROLLARY. *In Theorem* 10 *the hypothesis that there is convergence for* $u_s = \xi_s$, $s = 1, 2$ *can be replaced by the weaker condition that the set of numbers* $\{a_{ij}\xi_1^i \xi_2^j\}$ *is bounded.*

It is well known that Theorem 10 is no longer true if we use rectangular sum convergence. Consider for example the double power series

$$0! + (1!)u_2 + (2!)u_2^2 + \cdots + (n!)u_2^n + \cdots$$
$$- (0!)u_1 - (1!)u_1u_2 - (2!)u_1u_2^2 - \cdots - (n!)u_1u_2^n - \cdots$$

with all other terms zero. Using the rectangular sum definition, this series converges for $u_1 = u_2 = 1$, as the example of §1 shows. It also converges for $u_1 = 1$, u_2 arbitrary, and for u_1 arbitrary, $u_2 = 0$; but for no other values. This is seen from the following expression for the rectangular sum R_{sn}, $s \geqq 2$:

$$R_{sn} = (1 - u_1)\{0! + 1!u_2 + \cdots + n!u_2^n\}.$$

The reason for the failure of Theorem 10 for rectangular sum convergence can again be laid to non-absolute convergence. If $\sum a_{ij}u_1^i u_2^j$ converges *absolutely* for $u_s = \xi_s$ (by either definition), then the conclusion of Theorem 10 holds even for the rectangular sum definition.

An important power series derived from $\sum a_{ij}u_1^i u_2^j$ is obtained by taking all the u's equal. We thus obtain a *multiple power series in one variable*: $\sum a_{ij}t^{i+j}$. As in Theorem 10, we have

THEOREM 11. *If the multiple power series $\sum a_{ij}t^{i+j}$ in the variable t converges for $t=\xi$, (or if the set of numbers $\{a_{ij}\xi^{i+j}\}$ is bounded), then the series converges absolutely for all t in $|t|<|\xi|$, and converges uniformly in every closed region therein, thus representing an analytic function of t in $|t|<|\xi|$.*

Theorem 11 fails for convergence by rectangular sums.

There is another definition of convergence that is apropos for multiple power series in one variable. It is convergence by *circular sums*. A circular sum for the series $\sum a_{ij}$ is one of the form

$$C_n \equiv \sum a_{ij} \qquad (i+j \leqq n). \tag{2.9}$$

For the series $\sum a_{ij}t^{i+j}$,

$$C_n(t) = \sum_{s=0}^{n} b_s t^s \tag{2.10}$$

where

$$b_s = \sum a_{ij} \qquad (i+j=s). \tag{2.11}$$

In other words, the nth circular sum is precisely the nth partial sum for the *simple* power series to which $\sum a_{ij}t^{i+j}$ can be reduced.

According to Theorem 11, the region of convergence of the double power series $\sum a_{ij}t^{i+j}$ in the variable t is a circle. It is possible to express its radius in terms of the coefficients of the series. Let

$$\rho = \limsup |a_{ij}|^{1/(i+j)} \qquad (i+j \to \infty). \tag{2.12}$$

We then have

THEOREM 12. *The radius of convergence R of the series $\sum a_{ij}t^{i+j}$ is given by*

$$R = 1/\rho. \tag{2.13}$$

Case 1. $\rho<\infty$. Let R' be any positive number for which $R'<1/\rho$. Choose $\epsilon>0$ small enough so that $\theta\equiv R'(\rho+\epsilon)<1$. If $|t|\leqq R'$, then for all i, j with $i+j>N=N_\epsilon$,

$$|a_{ij}t^{i+j}| \leqq \theta^{i+j}, \qquad 0<\theta<1.$$

Now $\sum\theta^{i+j}$ converges (to the sum $\{1-\theta\}^{-2}$); whence by Theorem 4, series $\sum a_{ij}t^{i+j}$ converges absolutely for $|t|<R'$. But R' can be chosen arbitrarily in its given range. It follows that $R\geqq 1/\rho$.

Now suppose $|t|=\lambda>1/\rho$. On choosing $\epsilon>0$ small enough so that $\lambda(\rho-\epsilon)>1$, there will exist infinitely many sets (i, j) for which

$$|a_{ij}t^{i+j}| > [\lambda(\rho-\epsilon)]^{i+j}.$$

The series must therefore diverge, in virtue of Theorem 2. That is, $R \leqq 1/\rho$. Consequently, $R = 1/\rho$.

Case 2. $\rho = \infty$. It is readily shown by the method of Case 1 that the series diverges for every $t \neq 0$, so that $R = 0$.

Turning back to circular sums, we can assert that the radius of convergence of $\sum a_{ij}t^{i+j}$ by the circular sum definition is *at least as great as* R. For by the uniform convergence guaranteed by Theorem 11, the series can be reduced to a simple series, precisely the series arrived at by circular sums, convergent in $|t| < R$ at least. The following two examples show that the radius of convergence of the simple series *may* exceed R, but will not *always* exceed R.

Example 1. If $\sum a_{ij}t^{i+j}$ is the series whose first two rows are

$$1 + t + t^2 + \cdots + t^n + \cdots$$
$$+ t + t^2 + \cdots + t^{n+1} + \cdots$$

while all other terms are 0, then the circular sum definition gives the radius of convergence as 1, which is also the value of R.

Example 2. If $\sum a_{ij}t^{i+j}$ is the series having

$$1 + t + t^2 + \cdots + t^n + \cdots$$

as its first row and

$$1 - t - t^2 - \cdots - t^n - \cdots$$

as its first column, while all other terms are 0, then by circular sums the series converges to the sum 1 for all t, whereas $R = 1$.

For the series $\sum a_{ij}u_1^i u_2^j$ there is no result as precise as Theorem 12. We can, however, state

THEOREM 13. *Let* ρ *be given by* (2.12). *The series* $\sum a_{ij}u_1^i u_2^j$ *converges absolutely for all* u_1, u_2 *for which* $|u_s| < 1/\rho$, $s = 1, 2$, *and diverges for all* u_1, u_2 *for which* $|u_s| > 1/\rho$, $s = 1, 2$.

COROLLARY. *If* $\rho = \infty$, *every point* (u_1, u_2) *of convergence has at least one* $u_s = 0$.

COROLLARY. *Series* $\sum a_{ij}u_1^i u_2^j$ *converges absolutely for every* u_1, u_2 *for which*

$$\limsup \left| a_{ij}u_1^i u_2^j \right|^{1/(i+j)} < 1 \qquad (i + j \to \infty);$$

and diverges for every u_1, u_2 *for which the above superior limit exceeds one.*

The proof follows the line of demonstration of Theorem 12. Let

$$A(u_1, u_2) = \sum a_{ij}u_1^i u_2^j, \qquad B(u_1, u_2) = \sum b_{ij}u_1^i u_2^j \tag{2.14}$$

be two power series. The formal product is given by

$$C(u_1, u_2) = \sum c_{ij}u_1^i u_2^j \tag{2.15}$$

where

$$c_{mn} = \sum a_{ij}b_{rs}, \tag{2.16}$$

the sum in (2.16) being over all indices i, j, r, s for which, simultaneously,

$$i + r = m, \qquad j + s = n. \tag{2.17}$$

This suggests the following

DEFINITION. *By the Cauchy product series derived from the two series* $\sum a_{ij}$, $\sum b_{ij}$ *is meant the series* $\sum c_{ij}$ *where* c_{ij} *is given by* (2.16).

THEOREM 14. *If* $\sum a_{ij}, \sum b_{ij}$ *are absolutely convergent, then so is* $\sum c_{ij}$, *and*

$$\sum c_{ij} = (\sum a_{ij})(\sum b_{ij}). \tag{2.18}$$

Case 1. $a_{ij} \geqq 0$, $b_{ij} \geqq 0$. Let $A = \sum a_{ij}$, $B = \sum b_{ij}$. If σ is any σ-sum in the space of (m, n) (so that σ is a sum of form $\sum c_{mn}$), then $\sigma \leqq AB$. Hence $\{\sigma\}$ has a least upper bound C, and it is easy to see that $\sum c_{ij}$ converges to C and that $C = AB$. The theorem is thus true in this case.

Case 2. a_{ij}, b_{ij} *real.* Write $a_{ij} = a'_{ij} - a''_{ij}$ where $a'_{ij} = a_{ij}, a''_{ij} = 0$ if $a_{ij} \geqq 0$, and $a'_{ij} = 0$, $a''_{ij} = -a_{ij}$ if $a_{ij} < 0$; and similarly for the b's. Then (2.16) can be written

$$c_{mn} = \sum_{s=1}^{2} c_{mn}^{(s)} - \sum_{s=3}^{4} c_{mn}^{(s)}$$

where

$$c_{mn}^{(1)} = \sum a'_{ij}b'_{ij}, \qquad c_{mn}^{(2)} = \sum a''_{ij}b''_{ij}, \qquad c_{mn}^{(3)} = \sum a'_{ij}b''_{ij}, \qquad c_{mn}^{(4)} = \sum a''_{ij}b'_{ij}.$$

From Case 1 we now can write

$$C = C^{(1)} + C^{(2)} - C^{(3)} - C^{(4)},$$

where each $C^{(s)}$ is the sum of a convergent series of positive terms; and it is readily seen again that $C = AB$.

Case 3. a_{ij}, b_{ij} *complex.* We can split a_{ij} and b_{ij} into their real and imaginary components, and then apply the method of Case 2 to obtain the desired result.

3. Some remarks on k-tuply infinite series. It was stated in §1 that most of the work of §2 extends readily to the general case of the k-tuply infinite series $\sum a_{i_1 \cdots i_k}$ (or $\sum a_{(i)}$ for short). There are, however, certain points that may need further statement, and these we consider here.

In the definition of σ-sum, the rectangular sum R_{pq} of (2.1) is to be replaced by the k-dimensional rectangular sum

$$R_{p_1 \cdots p_k} = \sum a_{(i)} \qquad (0 \leqq i_s \leqq p_s,\ s = 1, \cdots, k); \tag{3.1}$$

and a σ-sum relative to the indices $(p_1, \cdots, p_k)$ will be written σ-sum $(p_1 \cdots, p_k)$. To extend Theorem 7 to k dimensions we use the following definition and lemma.

DEFINITION. *By a subseries of the* k*-tuple series* $\sum a_{(i)}$ *is meant any series obtained from* $\sum a_{(i)}$ *by fixing one or more of the* k *indices* $i_1, \cdots, i_k$.

LEMMA 5. *Let σ be a σ-sum for the series $\sum a_{(i)}$. Let r be any integer in the range $0<r<k$, and divide the indices $i_1, \cdots, i_k$ into two sets, one consisting of the indices $i_{m_1}, \cdots, i_{m_r}$, the other having the remaining $k-r$ indices. Write σ in the form*

$$\sigma = \sum_{(r)} \left\{ \sum_{(k-r)} a_{(i)} \right\}, \tag{3.2}$$

where the outer $\sum$ is summed over $i_{m_1}, \cdots, i_{m_r}$, and the inner $\sum$ over the remaining $k-r$ indices. Then the inner sum is a σ-sum for the $(k-r)$-tuple series obtained by fixing $i_{m_1}, \cdots, i_{m_r}$ in $\sum a_{(i)}$; and if its sum is denoted by $\sigma_{i_{m_1}, \cdots, i_{m_r}}$, then $\sum_{(r)} \sigma_{i_{m_1} \cdots i_{m_r}}$ is a σ-sum in the "space" of the indices $i_{m_1}, \cdots, i_{m_r}$.

This lemma is an immediate consequence of the meaning of "σ-sum."

We now state the extension of Theorem 7.

THEOREM 7′. *Let $\sum a_{(i)}$ be convergent to the sum A. Let r be any integer in the range $0<r<k$. If any r indices, say $i_{m_1}, \cdots, i_{m_r}$, are kept fixed the resulting $(k-r)$-tuple subseries converges; and if its sum is $A_{i_{m_1} \cdots i_{m_r}}$, then also the r-tuple series $\sum A_{i_{m_1} \cdots i_{m_r}}$ converges, and its sum is A.*

As this is the central theorem of the present work and as its proof is not too obviously an extension of the proof of Theorem 7, we now provide a proof.

The theorem is true for $k=1$ (vacuously) and for $k=2$ (Theorem 7). Assume it true through $k-1$; we shall prove it true for k. First suppose $r=1$. Since the σ-sum definition of convergence has no *preferred* index of summation, we may fix i_1.

Let $i_1=0$. To each $\epsilon>0$ there correspond indices $(p_1, \cdots, p_k)$ such that $|\sigma_1-\sigma_2|<\epsilon$ for every pair of σ-sums $(p_1, \cdots, p_k)$. Now let $\sigma_1^{(0)}, \sigma_2^{(0)}$ be any two σ-sums $(p_2, \cdots, p_k)$, taken from the $(k-1)$-tuple subseries for which $i_1=0$. If to each of $\sigma_1^{(0)}, \sigma_2^{(0)}$ we add those terms of $R_{p_1 \cdots p_k}$ (the same for both) not already present in them, we obtain σ-sums $(p_1, \cdots, p_k)$; call them σ_1, σ_2. From $|\sigma_1-\sigma_2|<\epsilon$ we get the relation $|\sigma_1^{(0)}-\sigma_2^{(0)}|<\epsilon$. Hence A_{i1} exists for $i_1=0$. If $i_1=1$, let $\sigma_1^{(1)}, \sigma_2^{(1)}$ be any two σ-sums $(p_2, \cdots, p_k)$, whose terms are the terms of $\sigma_1^{(0)}, \sigma_2^{(0)}$ plus the corresponding terms in the subseries for which $i_1=1$. We again get $|\sigma_1^{(1)}-\sigma_2^{(1)}|<\epsilon$. On setting $\sigma_j^{(1)}=\sigma_j^{(0)}+\sigma_j'^{(1)}$, $j=1, 2$, and recalling that $|\sigma_1^{(0)}-\sigma_2^{(0)}|<\epsilon$, we obtain the inequality $|\sigma_1'^{(1)}-\sigma_2'^{(1)}|<2\epsilon$. This shows that A_{i1} exists for $i_1=1$.

And so for every i_1. Hence, since any other i_s could have been used, all $(k-1)$-tuple series converge; and by our induction assumption, every subseries converges, since a subseries of $\sum a_{(i)}$ of fewer than $k-1$ indices is also a subseries of some $(k-1)$-tuple subseries.

There remains to establish the quantitative result of the theorem. Let $\epsilon>0$ be given. We are to show that indices $(p_1, \cdots, p_r)$ exist such that $|A-\sigma^*|<\epsilon$ for every σ^* that is a σ-sum $(p_1, \cdots, p_r)$ in the space of the indices $i_{m_1}, \cdots, i_{m_r}$; that is, σ^* is a σ-sum of form $\sum A_{i_{m_1} \cdots i_{m_r}}$, containing the rectangular sum $R_{p_1 \cdots p_r}=\sum A_{i_{m_1}, \cdots, i_{m_r}}$ $(0 \leq i_{m_s} \leq p_s,\ s=1, \cdots, r)$.

Now indices $(q_1, \cdots, q_k)$ exist so that

(a) $$|A-\sigma|<\frac{\epsilon}{2}$$

for every σ-sum $(q_1, \cdots, q_k)$ in the original space of $i_1, \cdots, i_k$. Choose the p's to have the values $p_s=q_{m_s}$, $s=1, \cdots, r$. Consider a fixed σ^* (a σ-sum $(p_1, \cdots, p_r)$). It will consist of a finite number, say g, of terms $A_{i_{m_1}\cdots i_{m_r}}$. For each such term there is a set of indices $(t_1, \cdots, t_{k-r})$ such that

(b) $$|A_{i_{m_1}\cdots i_{m_r}}-\sigma_{i_{m_1}\cdots i_{m_r}}|<\frac{\epsilon}{2g}$$

for every $\sigma_{i_{m_1}\cdots i_{m_r}}$ that is a σ-sum $(t_1, \cdots, t_{k-r})$. (This is so since each $A_{i_{m_1}\cdots i_{m_r}}$ is the sum of a convergent $(k-r)$-fold series.)

If we choose $u_s \geqq \max t_s$, $s=1, \cdots, k-r$, taken over all g sets $(t_1, \cdots, t_{k-r})$, then (b) holds uniformly for all g sets $(i_{m_1}, \cdots, i_{m_r})$ provided that each $\sigma_{i_{m_1}\cdots i_{m_r}}$ is a σ-sum $(u_1, \cdots, u_{k-r})$. We now impose on $u_1, \cdots, u_{k-r}$ the further conditions that each is at least as great as the corresponding index taken from $q_1, \cdots, q_k$ (minus, of course, $q_{m_1}, \cdots, q_{m_r}$). We have now arranged matters so that when suitably ordered, the k indices $u_1, \cdots, u_{k-r}, p_1, \cdots, p_r$ form a set $j_1, \cdots, j_k$ with $j_s \geqq q_s$, $s=1, \cdots, k$.

We may (and do) in particular, further restrict the $\sigma_{i_{m_1}\cdots i_{m_r}}$'s that appear in (b) to be rectangular sums (in the space of the $k-r$ indices other than $i_{m_1}, \cdots, i_{m_r}$), and all of them to have the same maximum indices $u_1, \cdots, u_{k-r}$ in that $(k-r)$-fold space. In other words two such rectangular sums are indistinguishable if one looks only at the $k-r$ indices of summation.

From (b) we obtain on summing,

$$\left|\sum_{(g)} A_{i_{m_1}\cdots i_{m_r}}-\sum_{(g)}\sigma_{i_{m_1}\cdots i_{m_r}}\right|<\frac{\epsilon}{2},$$

which is to say,

(c) $$\left|\sigma^*-\sum_{(g)}\sigma_{i_{m_1}\cdots i_{m_r}}\right|<\frac{\epsilon}{2}.$$

Now the conditions imposed on the $\sigma_{i_{m_1}\cdots i_{m_r}}$s insure us, as is easily seen, that $\sum_{(g)}\sigma_{i_{m_1}\cdots i_{m_r}}$, when expressed as a sum of terms $a_{i_1}\cdots i_k$ of the original series, is a σ-sum of that series; and the choice of $u_1, \cdots, u_{k-r}, p_1, \cdots, p_r$ further guarantees that it is a σ-sum $(q_1, \cdots, q_k)$. Hence

(d) $$\left|A-\sum_{(g)}\sigma_{i_{m_1}\cdots i_{m_r}}\right|<\frac{\epsilon}{2};$$

and this, in conjunction with (c), leads to

(e) $$|A-\sigma^*|<\epsilon.$$

As the choice of $(p_1, \cdots, p_r)$ depends only on ϵ, (e) asserts that $\sum A_{i_{m_1}\cdots i_{m_r}}$ converges to A, which is the desired conclusion.

MEAN DEVIATION OF THE BINOMIAL DISTRIBUTION

J. S. FRAME, Michigan State College

The binomial distribution is one of the topics usually studied in an elementary course in statistics. It is known that if in a single experiment the probability of success of an event is p and the probability of failure is $q=1-p$, then the probability of r successes and $n-r$ failures in n repetitions of the same experiment is the $(r+1)$th term in the binomial expansion of $(q+p)^n$, which we write as y_r^n:

$$y_r^n = \frac{n!}{r!(n-r)!} p^r q^{n-r}. \tag{1}$$

Now it is well known and easy to prove that the mean of this distribution is $\bar{x}=np$, and the standard deviation is $\sigma_n=\sqrt{npq}$, but a simple expression for the *mean deviation* (sometimes called "average deviation") *from the mean* seems to be not well known, and was not found in a search of elementary statistics books.

The mean deviation (MD_n) from the mean $\bar{x}$ is given by the equations:

$$MD_n = \sum_{r=0}^{n} |r - \bar{x}|\, y_r^n = \sum_{r=0}^{m} (np - r) y_r^n + \sum_{r=m+1}^{n} (r - np) y_r^n. \tag{2}$$

$$MD_n = 2\sum_{r=0}^{m} (np - r) y_r^n, \tag{3}$$

where $m=[np]$ is the largest integer not exceeding np. Experimental evidence shows that the values of MD_n seem to be related to the values of the largest term Y_n in the expansion of $(q+p)^n$.

For the symmetric case $p=q=1/2$, we have $\bar{x}=n/2$. Computation yields the following values for MD_n and Y_n:

$p=\frac{1}{2}$, $q=\frac{1}{2}$

n	1	2	3	4	5	6	7	8	9	10
MD_n	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{15}{16}$	$\frac{15}{16}$	$\frac{35}{32}$	$\frac{35}{32}$	$\frac{315}{256}$	$\frac{315}{256}$
Y_n	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{5}{16}$	$\frac{5}{16}$	$\frac{35}{128}$	$\frac{35}{128}$	$\frac{63}{256}$	$\frac{63}{256}$

We may verify for these values of n that

$$MD_n = \begin{cases} (n+1)Y_n/2 & \text{for } n \text{ odd}, \quad (p=q=1/2) \\ n\ Y_n/2 & \text{for } n \text{ even}, \quad (p=q=1/2), \end{cases} \tag{4}$$

but no simple formula seems to hold for both odd and even values of n.

For the case $p=2/3$, $q=1/3$, we have $\bar{x}=2n/3$. Computation yields the following values of MD_n and Y_n.

$p = \frac{2}{3}$, $q = \frac{1}{3}$

n	1	2	3	4	5	6	7	8	9	10
MD_n	$\frac{4}{9}$	$\frac{16}{27}$	$\frac{16}{27}$	$\frac{64}{81}$	$\frac{640}{729}$	$\frac{640}{729}$	$\frac{2240}{2187}$	$\frac{7168}{6561}$	$\frac{7168}{6561}$	$\frac{71680}{59049}$
Y_n	$\frac{2}{3}$	$\frac{4}{9}$	$\frac{4}{9}$	$\frac{32}{81}$	$\frac{80}{243}$	$\frac{80}{243}$	$\frac{672}{2187}$	$\frac{1792}{6561}$	$\frac{1792}{6561}$	$\frac{15360}{59049}$

We may verify for these values of n that

$$(5) \qquad MD_n = \begin{cases} 2(2n+1)Y_n/9 & \text{for } n = 3k+1, \quad (p = 2/3,\ q = 1/3) \\ 2(2n+2)Y_n/9 & \text{for } n = 3k+2, \quad (p = 2/3,\ q = 1/3) \\ 2(2n)Y_n/9 & \text{for } n = 3k, \quad (p = 2/3,\ q = 1/3), \end{cases}$$

but again we do not have a single simple formula for all values of n.

However, if we express MD_n as a multiple of Y_{n-1} instead of Y_n, then all cases come under a single simple formula:

$$(6) \qquad MD_n = 2npqY_{n-1},$$

where Y_{n-1} is the largest term in the expansion of $(q+p)^{n-1}$.

To prove formula (6) we shall first express each of the terms in (3) as the product of npq times the difference of two terms in the expansion of $(q+p)^{n-1}$. From the definition (1) we have

$$(7) \qquad (n-r)y_r^n = \frac{n!}{r!(n-1-r)!}p^r q^{n-r} = nqy_r^{n-1}, \qquad r \geqq 0.$$

$$(8) \qquad ry_r^n = \frac{n!}{(r-1)!(n-r)!}p^r q^{n-r} = npy_{r-1}^{n-1}, \qquad r \geqq 1.$$

From (7) and (8), using $p+q=1$, we have

$$(9) \qquad (np-r)y_r^n = p(n-r)y_r^n - qry_r^n = npq(y_r^{n-1} - y_{r-1}^{n-1}), \qquad r \geqq 1,$$

$$(9') \qquad (np-0)y_0^n = npqy_0^{n-1}.$$

Both sides of the equalities in (9) are positive for $r<np$, and negative for $r>np$, so the largest term Y_{n-1} in the expansion of $(q+p)^{n-1}$ is y_m^{n-1}, where $m=[np]$. (There are two equal largest terms if and only if np is an integer.) The sum of the members on the left side of (9) and (9′) for all r less than np is $MD_n/2$, by equation (3). But in evaluating the corresponding sum of the right hand members of (9) and (9′), we have a telescoping series in which all terms cancel except the single term $npq\,(y_m^{n-1})$, which equals $npqY_{n-1}$. This completes the proof of (6).

By using Stirling's formula for factorial n, namely

$$(10) \qquad n! = \sqrt{2\pi n}\,(n/e)^n e^{R(n)}, \qquad R(n) = \frac{1}{12n} - \frac{1}{360n^3} + \cdots,$$

it can be shown for large n that Y_{n-1} is approximately equal to $(2\pi npq)^{-1/2}$, or $1/\sqrt{2\pi}\sigma_n$. Thus from (6) we obtain the approximation

$$MD_n \sim \sqrt{2npq/\pi} = \sqrt{2/\pi}\,\sigma_n = 0.79788\sigma_n. \tag{11}$$

More exact computation, using the remainder terms in Stirling's formula, yields the better approximation

$$\frac{\pi}{2}(MD_n)^2 = npq + (np - [np])(nq - [nq]) - (1 - pq)/6 + E_n/24n, \tag{12}$$

where the error coefficient E_n becomes numerically less than or equal to unity as n becomes infinite, for all choices of np between 1 and $n-1$; and $[np]$ and $[nq]$ denote the greatest integers not exceeding np and nq respectively.

A SET OF EIGHT NUMBERS

ARTHUR PORGES, Western Military Academy

1. Introduction. In this paper the operation of adding the squared digits of any natural number A a finite number of times is proved to transform A either to unity or to one of a set of eight natural numbers closed under the operation.

2. Definitions. We use the expression *natural number* to denote a member of the set 1, 2, 3, $\cdots$ of positive integers. Zero has not been adjoined to this set and is not to be included in the definition.

The operator G is defined by the equation

$$G(A) = \sum_{i=1}^{R} X_i^2, \tag{1}$$

where A is a natural number of R digits given by

$$A = \sum_{i=1}^{R} X_i 10^{i-1}. \tag{2}$$

Since A has R digits, $X_R \neq 0$.

We note that $G(0) = 0$, and $G(1) = 1$.

Using the customary notation, we write $G^n(A)$, where $n > 1$, for n successive applications of the operator G to A.

G is not a linear operator since, in general, $G(A_1 + A_2) \neq G(A_1) + G(A_2)$.

The set of numbers

$$\begin{aligned} a_1 &= 4, & a_5 &= 89, \\ a_2 &= 16, & a_6 &= 145, \\ a_3 &= 37, & a_7 &= 42, \\ a_4 &= 58, & a_8 &= 20, \end{aligned} \tag{3}$$

is closed under the operation defined by (1). We call (3) *Set K*, and use the symbol a' to denote any non-specified element of the set. The equation

$$G^8(a') = a' \tag{4}$$

is easily verified.

Numbers of the form 10^n, $13 \cdot 10^n$, $10^{n+1}+3$, where n is a positive integer or zero, and others not specified here, satisfy the equation

$$G^r(A) = 1 \tag{5}$$

for some integer $r>0$. Any natural number satisfying (5) will be denoted by the symbol b'.

3. Preliminary Lemmas. In what follows, the symbols A and B always represent finite natural numbers in the denary system of notation.

LEMMA 1. *Any natural number A of R digits, where* $R \geqq 4$, *satisfies the inequality*

$$G(A) < A. \tag{6}$$

It is evident that $G(A) \leqq 81R$, and that $A \geqq 10^{R-1}$. The inequality

$$81R < 10^{R-1} \tag{7}$$

becomes, upon taking the common logarithm of each member and transposing,

$$\log_{10} R < R - 2.9085, \tag{8}$$

an inequality valid for $R \geqq 4$.

LEMMA 2. *For any natural number A there exists a positive integer n such that*

$$G^n(A) \leqq 162. \tag{9}$$

For $R \geqq 4$, Lemma 1 establishes the inequality (6). As a direct consequence of (6), the operator G applied to A a finite number of times must result in a natural number of less than four digits, since for $R=4$, $G(A) \leqq 324$.

For $R<4$, the following inequalities are readily established.

$$G(A) \leqq 243, \tag{10}$$

$$G^2(A) \leqq G(199) = 163, \tag{11}$$

$$G^3(A) \leqq G(99) = 162. \tag{12}$$

Since $G(A)$, where A is a three digit number, cannot exceed $3 \cdot 81 = 243$, (10) is obviously valid. Also, since $G(199) \geqq G(B)$ for any $B \leqq 243$, (11) holds. Finally, since $G(99) \geqq G(P)$ for any $P \leqq 163$, (12) is proved.

The inequalities (10), (11), and (12) complete the proof of Lemma 2.

4. Convergence of $G^n(A)$. The following theorem is the main result of this paper.

THEOREM 1. *For every natural number A there exists either a positive integer n such that* (5) *holds for all $r \geqq n$, or a positive integer m such that*

$$G^r(A) = a' \tag{13}$$

for all $r \geqq m$, where a' is some element of Set K.

From Lemma 2 it is evident we need prove the theorem only for $A \leqq 162$. The writer was unable to find a simple indirect proof sufficiently superior to the following direct one of selective verification to justify its inclusion here.

We consider two cases.

Case 1. $100 \leqq A \leqq 162$.

For A thus restricted, it is apparent that $G(A) \leqq G(159) = 107$. Direct application of the operator G to A over the range 100 to 107 gives

$$\begin{aligned} G(100) &= 1, & G^6(104) &= a' = 89, \\ G^2(101) &= a' = 4, & G^3(105) &= a' = 16, \\ G^5(102) &= a' = 89, & G(106) &= a' = 37, \\ G^2(103) &= 1, & G^5(107) &= a' = 89, \end{aligned} \tag{14}$$

thus completing the proof of the theorem for Case 1.

Case 2. $0 < A < 100$.

For $A = 10X + Y$, where $0 \leqq X \leqq 9$, and $0 \leqq Y \leqq 9$, the following identity is valid.

$$G(10X + Y) \equiv G(10Y + X). \tag{15}$$

Further, if $G^n(A) = a'$, and $G^m(B) = A$, it follows that there exists a number $h = n + m$ such that $G^h(B) = a'$.

By means of these considerations, it is possible to verify Theorem 1 numerically for all $A < 100$ by actual computation of $G^n(A)$ for 30 values of $A < 100$, thus completing the proof of the theorem.

The writer is aware of the inelegance of such a proof, and would like very much to see a simple indirect one. However, proving the non-existence of another set like (3), which seems a necessary step, is quite difficult because of the non-linear character of G.

COROLLARY. *For every natural number A there exists either a positive integer n such that $G^n(A) = 1$, or a positive integer m such that $G^m(A) = 4$.*

The corollary follows directly from Theorem 1 and the nature of Set K. Since every natural number is transformed either into unity or into an element of Set K by the operator G, we need only note that for every $a' \neq 4$, there exists a positive integer $r \leqq 7$ such that $G^r(a') = 4$.

THEOREM 2. *The number of digits N in $G(A)$, where A has R digits, satisfies the inequality*

$$N \leqq 2.9 + \log_{10} R. \tag{16}$$

This theorem is a simple consequence of the inequality $G(A) \leqq 81R$. We have

$$G(A) \leqq 10^{1.9} + \log_{10} R \tag{17}$$

a number of N digits, where $N \leqq 2.9 + \mathrm{Log}_{10} R$.

THEOREM 3. *The only solutions in natural numbers of*

$$G^n(A) = A,$$

where $n \geqq 1$, *are*

$$A = 1, \qquad n = J, \tag{19}$$

$$A = a', \qquad n = 8, \tag{20}$$

where J is any natural number.

If we assume the existence of a natural number $A > 1$ and different from a' such that $G^n(A) = A$ for some $n \geqq 1$, it follows that A would not be transformed into either unity or an element of Set K by a finite number of applications of the operator G to A. But this is a direct contradiction of Theorem 1, and hence the assumption is false.

5. Concluding Remarks. A problem suggested by the one just discussed is that of repeatedly summing the *cubed* digits of a natural number. A complication occurs, however, since there is more than one number A such that $H(A) = A$, where H is the operator analogous to (1) given by

$$H(A) = \sum_{i=1}^{R} X_i^3. \tag{21}$$

For example, $H(153) = 153$, $H(407) = 407$, and $H(371) = 371$. This destroys the factor of uniqueness, since $H(A)$ may be unity as when $A = 100$; or A may be transformed into a number A' like 153.

It is interesting to note that since for any number A transformed into some element of Set K by a finite number of applications of G we can construct a number $B = 10^A$ such that $G(B) = 1$, there are at least "as many" numbers satisfying (5) as (13). This intuitionally unsatisfying conclusion results from the comparison of two infinite sets.

Leibniz discovers the obvious. I have made some observations on prime numbers which, in my opinion, are of consequence for the perfection of the science of numbers . . . . If the sequence [of primes] were well known, it would enable us to uncover the mystery of numbers in general; but up till now it has seemed so bizarre that nobody has succeeded in finding any affirmative characteristic or property . . . . I believe I have found the right road for penetrating their [primes'] nature: but not having had the leisure to pursue it, I shall give you here a positive property, which seems to me curious and useful.—Leibniz, in a letter to the editor of the *Journal des Savans*, 1678. The discovery: a prime is necessarily of one or other of the forms $6n+1$, $6n+5$.—*Contributed.*

DISCUSSIONS AND NOTES

EDITED BY MARIE J. WEISS, Sophie Newcomb College, New Orleans 18, La.

The Department of Discussions and Notes is open to all forms of activity in college mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

ON PASCAL'S THEOREM

J. H. M. WEDDERBURN, Princeton University

Let $f(x, y, z)=0$ be the equation of a plane curve of degree n and P_i (x_i, y_i, z_i), $i=1, 2, \cdots, m$, the vertices of a closed polygon. The equation whose roots are the position ratios of the points in which the curve cuts the side P_iP_{i+1} has the form

$$t^n f(x_{i+1}, y_{i+1}, z_{i+1}) + \cdots + f(x_i, y_i, z_i) = 0$$

and the product of these roots is $(-1)^n f(x_i, y_i, z_i)/f(x_{i+1}, y_{i+1}, z_{i+1})$. Hence the product of the position ratios of all the points, real or imaginary, taken cyclically round the polygon is $(-1)^{mn}$; if the curve passes through a vertex, this is still true in the limit.

If we take $n=1$, that is, if the curve is a straight line, we see that *the cyclic product of the position ratios of the points in which a line cuts the sides of a closed polygon with m vertices is* $(-1)^m$. For the triangle this gives the theorem of Menelaus.

In preparation for the proof of Pascal's Theorem we take $m=3$, $n=2$ and the result may be stated as follows. *If a conic cuts the sides BC, CA, AB of a triangle in* $P_1, Q_1, P_2, Q_2, P_3, Q_3$ *and if the position ratios of these points relatively to the vertices of the sides on which they lie, taken in cyclic order, are* $p_1, q_1, p_2, q_2, p_3, q_3$, *then*

$$p_1q_1p_2q_2p_3q_3 = 1 \tag{1}$$

and conversely. The converse is immediate since a conic is uniquely determined by five points and (1) gives the position of the remaining point.

Let P_3Q_2, P_1Q_3, P_2Q_1 meet BC, CA, AB in R_1, R_2, R_3, respectively, and let the corresponding position ratios be r_1, r_2, r_3. By Menelaus' Theorem $r_1p_3q_2=-1$, $r_2p_1q_3=-1$, $r_3p_2q_1=-1$; multiplying corresponding sides of these equations together and using (1) we have $r_1r_2r_3=-1$ and hence R_1, R_2, R_3 are collinear, as required by Pascal's Theorem.

The following particular case is of some interest. *If* AP_1, BP_2, CP_3 *meet in a point S, then* AQ_1, BQ_2, CQ_3 *also meet in a point T.* For $p_1p_2p_3=1$ by Ceva's Theorem and hence also $q_1q_2q_3=1$. The point T can be readily constructed and a repetition of the construction, starting from T, leads back to S. If the conic is restricted to pass through two fixed points, this gives an interesting case of a Cremona transformation.

A metric proof of the lemma used in proving Pascal's Theorem is easily made for real intersections. If the conic is a circle, we have

$$AP_3 \cdot AQ_3 = AP_2 \cdot AQ_2, \qquad BP_1 \cdot BQ_1 = BP_3 \cdot BQ_3, \qquad CP_2 \cdot CQ_2 = CP_1 \cdot CQ_1$$

and multiplying corresponding sides of these equations together shows that the lemma is true for the circle; projection then gives it for any conic.

The proofs usually given for Menelaus' Theorem are metric, but it is easily shown that a change of projective coordinates merely multiplies the position ratio by a constant which depends on the given line and the base points and that for three lines forming the triangle of reference the product of the three constants is unity; and hence the theorem is projective.

SOME PROPERTIES OF THE LIMAÇON AND CARDIOID

J. H. BUTCHART, Grinnell College

Problem E 526 [1942, 404] of this MONTHLY, proposed by R. C. Yates, asked for the locus of a point P if the angles formed by the tangents from P to two fixed circles are equal. Yates thus called attention to a property of the circle of similitude.* This problem suggested the more difficult question: if tangents from P, one to each of two fixed circles, are allowed to vary while intersecting at a constant angle, what is the locus of P? The answer may be stated as follows:

THEOREM 1. *If two lines intersecting at a fixed angle are moved continuously tangent to two given circles, their intersection traces a limaçon whose double point lies on the circle of similitude of the two given circles.*

This locus is not new, for it is given as an exercise in Williamson's *Differential Calculus*, p. 372. However, since Williamson does not call attention to the location of the double point on the circle of similitude (S) of the given circles (O) and (O'), the following proof is believed to be new.

We might begin by assuming the variable tangents in an initial position, but it seems more convenient to use an indirect approach. Let C be a point on (S). Let Q be any point on the circle through $O'CO$, and let TP, $T'P'$ be tangents to (O), (O') parallel respectively to OQ, $O'Q$ and meeting CQ in P, P' such that these points are either both within CQ or else both without in the same direction, as in Fig. 1. Then since angles CPT and $CP'T'$ are constant, as they are equal to CQO and CQO' respectively, the segments QP and QP' are constant. When Q is in the special position diametrically opposite C, P and P' coincide, since then $CO:CT::CO':CT'$. Thus P and P' are identical in all positions and P describes a limaçon with double point at C.

That this discussion covers all possibilities may be seen by noting that the two tangents to (O) parallel to OQ meet the two tangents to (O') parallel to $O'Q$ in four points two of which lie on CQ and the other two on $C'Q$, where C' is the other point common to (S) and (OQO'). The two limaçons whose double points

* R. A. Johnson, Modern Geometry, p. 26.

are C and C' are evidently not in general congruent. If we were asking for the locus of all points from which tangents to the respective fixed circles can be drawn intersecting at a given angle, we should have to include two more limaçons symmetrical to these with respect to the line of centers. For completeness we should mention the cases where (S) is not a proper circle. If (O) and (O') are concentric, the locus degenerates to other circles concentric with them. When (O) and (O') are equal, (S) becomes their radical axis, but otherwise the discussion is unchanged.

The constant segment QP is less than, equal to, or greater than the diameter of (OQO') according as C lies without, on, or within both (O) and (O'). These conditions serve to classify the limaçons. The condition for a cardioid is equivalent to the requirement that the given angle formed by the tangents be the same as that at which the circles meet.

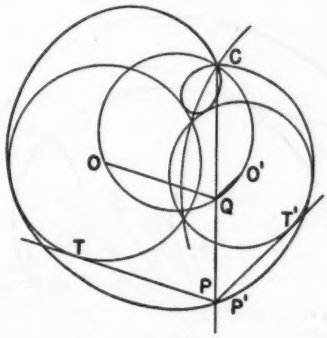


FIG. 1.

It is of interest to note that the triangle TCT' has a fixed shape. The truth of this follows easily from the evident similarity of triangles COT and $CO'T'$. It is well known that if a triangle, fixed in shape, has one vertex C fixed while another vertex T describes a circle, then the third vertex T' describes a circle.* We now note that the tangents to the two circles at T and T' meet on a limaçon with double point C.

Another comment which seems appropriate is that the intersection of tangents to two circles, the points of contact being collinear with one of the points common to the circles, traces a cardioid, for it is clear that the tangents meet at the same angle as the circles.

For the purpose of establishing the orthoptic† and some related properties of the cardioid synthetically, we may consider it an epicycloid generated by rolling the circle (R) on the fixed circle (F). This and the corresponding theorem for the general limaçon are given as an exercise in Williamson's *Differential Calculus*, p. 350. The reader may be interested in the following simple proof that

* N. A. Court, College Geometry, 1925, p. 48.

† R. C. Archibald, Encyclopedia Britannica, 14th edition, vol. 6, p. 897.

the limaçon can be generated as an epitrochoid. Let (R) carry P, the point which describes the curve. Let (R) start rolling with RP extended towards F, and take O on this fixed diameter of (F) so that FO equals PR. When (R) has rolled to a general position, draw PO and extend it to meet at Q the circle with center F and radius FO. Then $FRPQ$ is a parallelogram and QP is constant, thus showing that P moves along a limaçon with double point O. If P lies on (R), it will trace the cardioid. A definitive property of the limaçon is that as FR turns through an arbitrary angle, RP turns through an angle twice as large. It is also clear in this connection that the normal passes through the point of contact between (F) and (R) and that the line joining this point C to P will, for the cardioid, turn through an angle half again as great as the angle described by FC. It is well known that the evolute of the cardioid is another cardioid homothetic to it with respect to the center in the ratio $-1:3$.* We shall now establish the following:

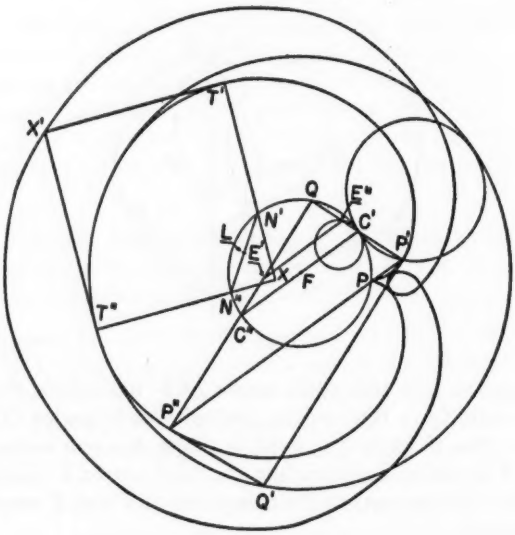


FIG. 2.

THEOREM 2. *The perpendicular tangents to a cardioid at the ends of a cuspidal chord meet on a circle concentric with* (F) *having a radius three times that of* (F), *and the normals at these points meet on* (F) *itself.*

Consider the rolling circle in two positions touching (F) at C' and C'' which are ends of a diameter of (F). (See Fig. 2.) Then the corresponding points P' and P'' of the cardioid are the reflections of the cusp P across the tangents at C' and C''. Since $C'P'$ and $C''P''$ are also reflections of $C'P$ and $C''P$ across $C'C''$, they meet orthogonally at Q on (F). This establishes the part of the theo-

* For a synthetic proof of this, see my solution of problem E 600 [1944, 587], this MONTHLY.

rem concerning the normals. By the argument establishing the evolute, we see that $C'P'$ and $C''P''$ touch the evolute at E'' and E' respectively such that $C'P'$ is three times $E''C'$ and $C''P''$ is three times $E'C''$. Thus from Q as homothetic center, three times $E''E'$ equals $P'P''$, and $E'E''$ is homothetic to $P'P''$ in the ratio $-1:3$ with respect to the center F. Thus $E'E''$ passes through the cusp of the evolute, and by homothetic figures the first part of the theorem is proved.

A right angle may also have its sides tangent to a cardioid without the points of contact T', T'' being collinear with the cusp. A characteristic feature of this situation is that the normals at T', T'' cut (F) at points N', N'' which are 60° apart. This is clear when we note that as the point of contact N between (R) and (F) moves through the arc θ, NT turns through the angle $3\theta/2$. Thus, in order for the normals to meet at 90° or 270°, it is necessary for the points of contact between (F) and the two positions of (R) to be 60° or 180° apart. The latter has been discussed when we were considering C', C'' above, so we know that N' and N'' are 60° apart. We shall proceed to the proof of the following:

THEOREM 3. *Perpendicular normals to a cardioid at points not collinear with the cusp meet on a nodal limaçon, and the perpendicular tangents at the same points meet on a similar limaçon three times as large.*

We see at once that $N'T'$ and $N''T''$ meet at a point X which is on the circle (L) constructed on $N'N''$ as diameter. As L turns through the angle θ about F, it may be shown that LX turns through the angle 2θ. For while N' is describing the arc θ on (F), since $FN'N''$ is equilateral, it is also moving to $\overline{N}$ on (L) through the arc θ. At the same time X is moving on (L) through the arc 2θ to $\overline{X}$, for $\overline{N}\overline{X}$ makes the angle $3\theta/2$ with a parallel through $\overline{N}$ to $N'X$. Since LX turns through the angle 2θ while FL is turning through θ, X traces a limaçon. That it is nodal appears from the fact that LX is greater than $LF/2$, the radius of the circle centered at L rolling on an equal circle centered at F so as to carry X. The second part of the theorem follows from the homothetic relationship between the evolute and the cardioid itself.

Mathematics for/by the layman. It was with this bold axiom that the pamphlet began, and written in the form of a geometrical theorem it proceeded to prove the impossibility of the existence of God. It ended triumphantly with the three letters Q.E.D., quod erat demonstrandum. To Shelley, who understood nothing of mathematics, this formula had always seemed like a magician's spell for the evocation of Truth—André Maurois, *Ariel*, p. 34.

I could prove God statistically. Take the human body—the chance that all functions of the individual would just happen is a statistical monstrosity.—George Gallup, *Reader's Digest*, October, 1943.

The improbable is also a part of probability—Aristotle.—*E. D. Schell.*

CLUBS AND ALLIED ACTIVITIES

EDITED BY J. S. FRAME

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to J. S. Frame, Michigan State College, East Lansing, Michigan.

CLUB REPORTS 1944-45

Junior Mathematical Club, University of Chicago

During the past year the following papers were presented by students:

Statistics, by Herman Rubin

One and two-dimensional intuitive topology, by R. B. Leipnik

Symbolic logic, by R. P. Brady

A class of definite boundary value problems, presented by Hyman Zimmerberg to the Senior Mathematical Club. As a prize for the best student paper in content and presentation, Mr. Zimmerberg was awarded a copy of Ince's *Ordinary Differential Equations*.

In addition to the student papers, the following talks were presented.

A history of mathematics in America before 1900, by Professor R. G. Sanger

Elliptic integrals, by Dr. J. E. Wilkins, Metallurgical Laboratory, University of Chicago

An application of tensor analysis to moving axes in Newtonian mechanics, by George Platzman, Institute of Meteorology, University of Chicago

Mathematical aspects of the structure of music, by Professor Emeritus A. C. Lunn

Fundamental aspects of valuation theory, by Professor O. F. G. Schilling

Mathematical biophysics of visual aesthetics, by Professor N. Rashevsky, Department of Mathematical Biophysics, University of Chicago

Polynomials with real coefficients whose zeros have negative real parts, by Professor H. S. Wall, Illinois Institute of Technology

Rational approximations to irrational numbers, by Professor L. R. Ford, Illinois Institute of Technology

A functional equation of differential type, by Professor M. L. Hartung

Some applications of the Poisson summation formula, by Dr. W. Karush, Metallurgical Laboratory, University of Chicago.

In addition to the teas given before the talks, the club sponsored parties in both the Autumn and Winter Quarters and a picnic in the Spring Quarter. The officers for the year 1944–45 were: President, Charles Nichols, who was succeeded by Hyman J. Zimmerberg on January 10, 1945; Social Chairman, Marie A. Wurster; Treasurer: Betty Alexander; Committee: Verna La Mantia, Herman Rubin, Harley Flanders. The officers elected for 1945–46 are: President, Marie A. Wurster; Social Chairman, Sally Springer; Treasurer: Flora Dinkines; Committee: August Newlander, Taffee Tanimoto, Dorothy Strayhorne.

Mathematics Club, Oberlin College

During the year from November through May, the following talks were given:

Unsolved mathematical problems, by Professor E. P. Vance at an informal meeting at the home of Professor Carr.

Three unsolvable problems in geometry, by Jeremiah Howald, Donald Taub, and Robert Stuckert

Calculating machines, by Sarah-Lou Lotz

History of mathematics clubs, by Mary Kinsman

A solution of a problem in the MONTHLY, by John Hofmann

The algebra of sets, by Charlotte Peters

Absolute values, by Helen Cutler and Joanne Benton

Logical paradoxes, by Professor Garvin of the Philosophy Department

The principle of duality, by Rodney Hood, who won a prize for this talk

Quaternions, by Ruth Cheney

The rule of double false position, by Margaret Waugh

A problem in limits, by Paul Meier.

We had a Christmas Party at which we had mathematical relays and games. We ended the year with a banquet at the Oberlin Inn, followed by a talk entitled:

The calculus of variations, by Professor E. J. Mickle of Ohio State University.

The officers for 1944–45 have been: President, Ruth Cheney; Vice-President, Rodney Hood; Secretary-Treasurer, Mary Kinsman; Social Chairman, Sarah-Lou Lotz; Publicity Chairman, Norman Weinstein; and Faculty Adviser, Professor E. P. Vance. The officers for 1945–46 are: President, Rodney Hood; Vice-President, Mary Kinsman; Secretary-Treasurer, Charlotte Peters; Social Chairman, Margaret Waugh; Publicity Chairman, Jeremiah Howald; and Faculty Adviser, Professor John Randolph.

Mathematics Club, College of Wooster

The Mathematics Club opened its meetings this year with a picnic, and plans to close them with another picnic.

During the course of the year at bi-monthly meetings invited speakers from various departments of the college told in what ways the study of mathematics was important to their respective fields. Their titles were as follows:

Mathematics in geology, by Dr. Karl Ver Steeg

Use of mathematical principles in physics, by Dr. Earl Ford

Logical aspects of mathematics, by Dr. John Hutchison

Roman engineering, by Dr. Frank Cowles

Mathematics and poets, by Dr. Lowell Coolidge

Mathematical education in Scotland, by Mrs. Haldean Lindsey

Mathematics and music, by Professor Daniel Parmelee.

The officers arranging the program this year are: President, Anne Widener; Secretary-Treasurer, Lois Hayenga. The Faculty Adviser is Dr. C. O. Williamson.

Kappa Mu Epsilon, Albion College

Even under the present circumstances the activities of this year have not diverged considerably from those of normal times. The six new members initiated in March were Phyllis Beckman, Corinne Calkins, Ruth Helzer, Audrey Schuett, Mary Shattuck, and Amy Thomas.

Responses such as algebraic formulae, early mathematicians, geometric theorems, and unusual applications of mathematics, were given in answer to roll call.

Papers presented during the year included:

Lesser known application of mathematics and applications of determinants, by Virginia Tripp

Isolation and approximation of roots, by Audrey McPherson

Mathematics in economics, by George Kawano

Mathematics as a recreation, by Harriette Leonard

Trisecting an angle, by Fusajiro Aburano

Teaching of secondary mathematics, by Janis Barker and Marion Bunte

Applications of elementary mathematics, by Charles Parkhurst

Probabilities, by Beryll Voelker

Fourier series, by Susanne Porter

Application of mathematics to statistics, by Mrs. MariAnn Jones.

Officers for the year 1944–45 were: President, Virginia Tripp; Vice-President, Fusajiro Aburano; Secretary-Treasurer, Marion Bunte, and an additional member of the Program Committee, Harriette Leonard. Faculty Advisers were Professor E. R. Sleight and Assistant Professor E. E. Ingalls.

Kappa Mu Epsilon, Upsala College

Nine meetings were held. Three of these featured papers by students, namely:

Approximation of roots by short-cut methods, by Mrs. Mary McKim

Mathematics in art, by Audrey Richter

A comparison between the Ptolemaic and Copernican systems of the universe, by Betty Rudebock.

Two meetings were devoted to applications as discussed by two engineers from the Wright Aeronautical Corporation:

Gears, by Mr. Jos. Rice, and

Numerical calculations by aid of trigonometry, by Mr. J. Rudebock.

At our initiation banquet, Professor Virgil Mallory of Montclair State Teachers College discussed

The prospects of mathematics in the post-war era.

The officers for the coming year are: President, Audrey Richter; Vice-President, Mrs. Mary McKim; Secretary, Natalie Manno; Treasurer, Marion Larsen; Historian, Betty Rudebock; Faculty Adviser and Corresponding Secretary, Professor M. A. Nordgaard.

RECENT PUBLICATIONS

EDITED BY H. P. EVANS, University of Wisconsin

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y., and not to any of the other editors or officers of the Association.

Empirical Equations and Nomography. By D. S. Davis. New York and London, McGraw-Hill Book Company, Inc., 1943. 200 pages. $2.50.

The two parts of the book, Part I, Empirical Equations, and Part II, Nomography, may be read independently. The mathematical background required from the reader does not go beyond elementary algebra and geometry but requires some mathematical maturity. Part I deals mainly with the problem of fitting formulas to empirically known relationships between two variables. One chapter is devoted to the same problem for three variables. In view of the elementary approach, no attempt is made to treat the problem statistically, and only one method is presented in detail: a number of observed pairs of values are plotted and, if the relationship suggested by the graph does not appear linear, a change of variables is found which transforms this graph into a linear one; then a straight line is fitted by the method of averages. The method of least squares for fitting a straight line is briefly mentioned, while the possibility of fitting a curve to the original data, without previous rectification, is not considered. Interpolation by the Lagrange-formula is described and illustrated in some detail; no mention is made of interpolation by differences.

Part II contains the derivations and very careful descriptions of procedures used to construct nomographs for relationships of the types (A) $f(x)=F(y)+\phi(z)$, (A′) $1/f(x)=1/F(y)+1/\phi(z)$, (B) $f(x)=F(y)\cdot\phi(z)$ (logarithmic charts), (C) $f(x)=\psi(x)\cdot F(y)+\phi(z)$, (D) $f(x)=F(y)/\phi(z)$ (non-logarithmic charts), and for various combinations of those types. A concluding chapter describes the construction of slide rules for special purposes.

The most impressive feature of the book is the great number of practical examples illustrating a great variety of possible situations, as well as the wealth of problems at the end of each chapter. Most of the examples and problems are taken from the field of industrial chemistry, which may make the book particularly interesting to readers engaged in this field. Without attempting an exhaustive mathematical treatment of either of the topics presented in Part I and Part II, the author has succeeded in giving a clear and interesting presentation of a considerable number of elementary methods which should prove useful in the hands of practical men, mainly in the fields of engineering and industrial management.

Z. W. BIRNBAUM

Nomography-Probability-Complex Functions. Three lectures by L. R. Ford, A. H. Copeland, Emil Artin. Notre Dame, Indiana (Notre Dame Mathematical Lectures, Number 4), University Press, Notre Dame (lithoprinted), 1944. 70 pages. $1.25.

While this booklet is listed under the above title in the Notre Dame series, it actually consists of four papers, whose full titles and numbers of pages are given below. The paper by Menger on probability and statistics is an appendix to Copeland's lecture. The three lectures together with a lecture by Menger on the algebra of analysis (which in expanded form has been published as Number 3 of the series) constituted a symposium on the presentation of certain topics of advanced college mathematics, arranged by the University of Notre Dame in April, 1943.

Alignment charts. By L. R. Ford. 28 pages. The exposition is so skillful that it will satisfy the mathematician's desire for generality and yet be intelligible to an undergraduate. The theory is divided into three parts with the italicized titles: (1) *Scales.* A scale is "a curve with certain marks upon it to which numbers are attached." If the variable t denotes the scale-reading, it is convenient to introduce rectangular coordinates and take the equations of the curve in the parametric form $x=f(t)$, $y=g(t)$. (2) *Third order determinants.* An alignment chart consists of three scales, say $x_i=f_i(t_i)$, $y_i=g_i(t_i)$, $i=1, 2, 3$, across which a straight line is laid, its intercepts on the three scales determining a functional relation among t_1, t_2, t_3. Necessary and sufficient conditions that a given functional relation among three variables be reproducible by an alignment chart are given in terms of the vanishing of certain third order determinants. (3) *Transformation theory.* The general projective transformation of the plane is introduced analytically as a tool for transforming alignment charts. As an illustration it is shown how any desired part of an alignment chart lying in a convex quadrilateral may be transformed so as to occupy a given rectangle.

The teaching of the calculus of probability. By A. H. Copeland. 12 pages. This outline of the calculus of probability begins with sequences and expected values rather than measure and probabilities. It is assumed that if measurements $x^{(1)}, x^{(2)}, \cdots, x^{(n)}$ are made on a physical quantity, then these are the first n terms of an infinite sequence $x=\{x^{(i)}\}$ having certain properties, one of which is that

$$\lim_{n\to\infty} \sum_{i=1}^{n} x^{(i)}/n$$

exists. This is called the expected value of the quantity measured and is written $p(x)$. Given a (real-valued) function of real variables, a corresponding (sequence-valued) function of sequences is defined, for example, the function $f(x, y)$ of sequences $x=\{x^{(i)}\}$ and $y=\{y^{(i)}\}$ is defined to be the sequence $\{f(x^{(i)}, y^{(i)})\}$. Constant functions c are identified with sequences $c, c, c, \cdots$. From this it

follows that the expected value operator p is a linear operator. Probabilities are then introduced as expected values in sequences of ones and zeros corresponding respectively to successes and failures of an event. It is indicated how from this basis may be developed the usual tools of probability theory——distribution functions, Stieltjes integrals, *etc.* The lecture concludes by raising the following important problem of application of the theory: From a knowledge of a finite number of terms $x^{(1)}, x^{(2)}, \cdots, x^{(n)}$ of a sequence x of zeros and ones, representing the outcome of n trials of an event, how can the hypothesis that the probability $p(x)$ of the event has a given value, say $\frac{1}{2}$, be verified? The reviewer found the author's treatment of this well-known problem of statistical inference obscure, but the difficulty may lie in the highly condensed nature of the exposition.

The author's approach to probability theory by way of sequences of observations on physical quantities has the pedagogical advantage of seeming close to the material to which the theory is to be applied. The danger of this approach in its present form is that it leaves the impression that every series of physical measurements is in "statistical control" and hence a fit subject for statistical inferences valid for random samples. We owe to Shewhart our skepticism of this assumption. The usual way of avoiding the difficulty is to develop a model more independently of series of physical measurements and then to emphasize that not all series of measurements fit the model.

On the relation between calculus of probability and statistics. By Karl Menger. 10 pages. The editor considers the problem mentioned above of testing the hypothesis that the probability of an event has a given value. He uses this example to illustrate some of the basic concepts of the Neyman-Pearson theory of testing statistical hypotheses. The reader is gracefully introduced to these basic ideas without being burdened with the specialized terminology of the theory.

On the theory of complex functions. By Emil Artin. 14 pages. The purpose of the lecture is to indicate how a course in complex variables may be simplified by a heavier use than customary of certain simple topological concepts. A function $V(A, \zeta)$ of arcs A and points ζ is defined for pairs A, ζ with ζ not on A as the increase of the argument of the vector from ζ to a point z on the arc A as z sweeps out the (oriented) arc. Chains C of arcs are introduced and $V(C, \zeta)$ for ρ not on C is defined additively over the arcs of the chain. It is noted that for a closed chain, $V(C, \zeta)/(2\pi)$ is an integer-valued function, constant for ζ on any connected open set disjoint from C, and this function is called the *winding number* $W(C, \zeta)$. It is used extensively in developing the theory; for example, Cauchy's theorem reads as follows: If $f(z)$ is analytic in the open set D, and if C is a closed chain of rectifiable arcs in D for which the winding number $W(C, \zeta)$ vanishes for every ζ in the complement of D, then $\int_C f(z)dz = 0$. The developments extend through Cauchy's integral formula to some of the more elementary theorems concerning zeros and isolated singularities of analytic functions.

HENRY SCHEFFÉ

NEW BOOKS RECEIVED

Cambridge Joint Advisory Committee for Mathematics: Syllabuses for Examinations Taken by Sixth Form Pupils. Cambridge University Press, 1945. 12 pages. 6 d.

Collected Works. Vol. 1. By P. L. Tchebychev. Moscow-Leningrad, Academy of Sciences of the U.S.S.R., 1944. 342 pages.

Curso de Análisis Matemático. Vol. 1. By Cristóbal de Losada y Puga. Lima, Universidad Catolica del Peru, 1945. 15+632 pages.

Elementary Statistics. By Hyman Levy and E. E. Preidel. New York, Ronald Press Co., 1945. 7+184 pages. $2.25.

Engineering Preview. An Introduction to Engineering Including the Necessary Review of Science and Mathematics. By L. E. Grinter, H. N. Holmes, H. C. Spencer, Rufus Oldenburger, Charles Harris, R. G. Kloeffler and V. M. Faires, New York, Macmillan Co., 1945. 10+581 pages. $4.50.

Fermagoric Triangles. By Pedro Pizá. (Publication No. 1 of the Polytechnic Institute of Puerto Rico.) San Germán, P.R. 153 pages. $3.00.

How to Solve It. By G. Polya. Princeton, University Press, 1945. 15+204 pages. $2.50.

Lobachevsky. By V. F. Kagan. Moscow-Leningrad, Academy of Sciences of the U.S.S.R., 1944. 348 pages (In Russian).

Mathematics of Finance. By T. E. Raiford. Boston, Ginn and Co., 1945. Without Tables: 3+176 pages, $2.50; with tables: 3+176+305 pages, $3.60.

Plane and Spherical Trigonometry. By F. M. Morgan. New York, American Book Co., 1945. 15+257+72 pages. $2.50.

Select Topics of Plane Analytic Geometry for Scientific and Technical Workers. By M. W. Rosanoff. Brooklyn, Long Island University Press, 1944. 8+76 pages. $1.25.

The Meaning of Relativity. Second Edition. By Albert Einstein. Princeton, University Press, 1945. 4+135 pages. $2.00.

The Simple Calculation of Electrical Transients. By G. W. Carter. Cambridge, University Press; New York, Macmillan Co., 1945. 8+120 pages. $1.75.

Theory of Functions. Part 1. By Konrad Knopp. (Translated by Frederick Bogehmihl from the fifth German edition.) New York, Dover, 1945. 7+146 pages. $1.25.

Precision, morals, torches, and algebra. My thoughts were directed to the problem of how the moral sciences may be reduced to mathematical precision.—Christian Wolff (1716).

I, at any rate, have not the same lively hope as Condorcet, or even Edgeworth, "to illuminate the moral and political Sciences by the torch of Algebra."—J. M. Keynes (1921).—*Contributed.*

Send interesting short notes to Professor E. T. Bell, California Institute of Technology, Pasadena 4, California.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND HOWARD EVES

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to Howard Eves, College of Puget Sound, Tacoma 6, Washington.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 681. *Proposed by W. B. Campbell, Philadelphia*

From *Mrs. Miniver*: "She saw every relationship as a pair of intersecting circles. The more they overlapped, it would seem at first glance, the better the relationship; but this is not so. Beyond a certain point, the law of diminishing returns sets in, and there are not enough private resources left on either side to enrich the life that is shared. Probably perfection is reached when the area of the two outer crescents, added together, is exactly equal to that of the leaf shaped piece in the middle. On paper there must be some neat mathematical formula for arriving at this; in life, none."

Discuss the possibility of a unique solution for circles of given radii.

E 682. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Find three positive integers x, y, z, such that the sum of ratios

$$\frac{y}{z}+\frac{z}{x}+\frac{x}{y}$$

is an integer. (Hint: suppose $x+y+z=23$.)

E 683. *Proposed by Irving Kaplansky, Columbia University*

A person tosses a coin n times. For every unbroken run of i heads he will receive 2^i dollars. What is his expectation?

E 684. *Proposed by Paul Erdös, Stanford University*

Prove that $n!^{(n-1)!}$ divides $(n!)!$; and that if n is not a power of a prime, $n!^{n(n-2)!}$ divides $(n!)!$.

E 685. *Proposed by Peter Scherk, University of Saskatchewan*

Prove that the equation $x^r-1=p^n$, where p is prime while r and n are integers greater than 1, has only one solution. (*Cf.* E 663 [1945, 159].)

SOLUTIONS

Four Equal Spheres in a Tetrahedron

E 637 [1944, 472]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Locate a plane which touches four equal spheres inscribed in the trihedra at the respective vertices of a given tetrahedron.

Solution by Howard Eves, College of Puget Sound. There are four planes of the required type, one parallel to each face of the tetrahedron. A representative one of these planes may be found as follows. Let h be an altitude and r the inradius of the tetrahedron. By simple considerations of similarity, the required plane parallel to the face corresponding to h is located at distance h' from the opposite vertex, here $h':h=h:(h+2r)$, so that $h'=h^2/(h+2r)$.

The analogous problem for a triangle is solved similarly.

A Property of Central Conics

E 645 [1944, 530]. *Proposed by C. D. Olds, Purdue University*

Let P_1, P_2, P_3 be any three points on a plane curve C, and let O be any point in the plane of C. Show that if the areas of the three triangles OP_2P_3, OP_3P_1, OP_1P_2 are connected by a relation independent of the coordinates of P_1, P_2, P_3, then C is either a straight line or a conic with center O.

Solution by the Proposer. Let O be the origin of coordinates, and let P_i be the point (x_i, y_i). Then the respective areas a_1, a_2, a_3 are given by

$$\pm 2a_1 = y_2x_3 - y_3x_2, \qquad \pm 2a_2 = y_3x_1 - y_1x_3, \qquad \pm 2a_3 = y_1x_2 - y_2x_1,$$

which expressions are functions of x_1, x_2, x_3. We think of y as a given function of x for points on C.

Since a_1, a_2, a_3 must satisfy a relation independent of x_1, x_2, x_3, the vanishing of the Jacobian $\partial(a_1, a_2, a_3)/\partial(x_1, x_2, x_3)$ leads to the equation

$$y_3'(y_3k + x_3l) + x_3m + y_3l = 0, \tag{1}$$

where

$$k = x_1x_2(y_2' - y_1') - (x_2y_1 - x_1y_2),$$
$$l = x_2y_1y_1' - x_1y_2y_2',$$
$$m = y_1y_2(y_2' - y_1') - y_1'y_2'(x_2y_1 - x_1y_2),$$

and $y_1' = \partial y_1/\partial x_1$, *etc.* Notice that k, l, m are functions of x_1, x_2 only. The following two cases present themselves.

(a) Suppose that $k=l=m=0$. From $k=0$ and $m=0$, eliminate the expression $x_2y_1-x_1y_2$, getting

$$(y_2' - y_1')(y_1y_2 - y_1'y_2'x_1x_2) = 0,$$

so that either $y_2'=y_1'$ or $y_1y_2=y_1'y_2\cdot y_2'y_1$. If $y_2'=y_1'$, then $m=0$ implies that $x_2y_1=x_1y_2$. For a given P_1, the locus of P_2 is the straight line $y_1x=x_1y$. On the other hand, if $y_1y_2=y_1'x_2\cdot y_2'x_1$, then combining this with $l=0$ we get $y_2^2=y_1'^2x_2^2$, or

$$\pm x_2 y_1' - y_2 = 0.$$

For any definite constant value of y_1', either of these equations gives a straight line.

(b) If k, l, m are not all identically zero, then (1) gives, when integrated,

$$y_3^2 k + 2y_3 x_3 l + x_3^2 m = f(x_1, x_2).$$

Thus, for arbitrary values of x_1, x_2, the locus of (x_3, y_3) is a conic with its center at O.

Weighed and Found Wanting

E 651 [1945, 42]. *Proposed by E. D. Schell, Arlington, Virginia*

You have eight similar coins and a beam balance. At most one coin is counterfeit and hence underweight. How can you determine whether there is an underweight coin, and if so, which one, using the balance only twice?

Solution by Monte Dernham, San Francisco. Weigh three coins against any other three. If they balance, weigh the seventh coin against the eighth. If the two sets of three do not balance, weigh against each other any two coins of the lighter set. If these two balance, the remaining one of that set is counterfeit.

Also solved by D. W. Alling, Murray Barbour, Ruth Beck, D. H. Browne, W. E. Buker, W. E. Bunyan, H. N. Carleton, Howard Eves, Orrin Frink, Jr., R. E. Greenwood, J. E. Hanson, Robert Hoskins, Irving Kaplansky, V. L. Klee, Jr., H. D. Larsen, A. D. M. Lewis, J. S. Miller, G. H. Neugebauer, John Pack, W. O. Pennell, J. Price, Jacob Schachter, E. P. Starke, R. H. Urbano, Jeanette Van Os, W. R. Van Voorhis, Alan Wayne, T. L. Woolard, and the proposer.

The following generalization was noticed by Kaplansky, Neugebauer, and Pennell. In any set of N coins, where $3^{n-1} \leqq N < 3^n$, not more than n balancings are needed to determine whether there is an underweight coin, and if so, which one. But if there is known to be an underweight counterfeit among N coins, where $3^{n-1} < N \leqq 3^n$, then not more than n balancings are needed to determine the counterfeit.

An Extension of E 569

E 653 [1945, 42]. *Proposed by J. H. Butchart, Grinnell College*

The ends of a chord UV of the circle $r = a$ have the parametric angles ϕ and $k\phi$, where k is a constant greater than 1. Show that the locus of the midpoint of UV is a prolate epitrochoid whose polar equation is

$$r = a \cos \frac{k-1}{k+1} \theta.$$

Show also that the envelope of UV is an epicycloid, the point of contact dividing UV in the ratio $1:k$.

Solution by R. H. Urbano, Albany, N. Y. We have

$$r = a\cos\tfrac{1}{2}(k\phi - \phi), \qquad \theta = \tfrac{1}{2}(k\phi - \phi) + \phi = \tfrac{1}{2}(k\phi + \phi).$$

Therefore

$$r = a\cos\frac{k-1}{k+1}\theta.$$

The line UV, joining $(a\cos\phi, a\sin\phi)$ and $(a\cos k\phi, a\sin k\phi)$, has the equation $F=0$, where

$$F = (x - a\cos\phi)(\sin\phi - \sin k\phi) - (y - a\sin\phi)(\cos\phi - \cos k\phi).$$

The envelope of this line is the simultaneous solution in x and y of the equations $F=0$ and $\partial F/\partial\phi = 0$. Since

$$\frac{\partial F}{\partial\phi} = x(\cos\phi - \sin k\phi) + y(\sin\phi - k\sin\phi) + a(k-1)\cos(k-1)\phi,$$

the result is

$$x = a(\cos k\phi + k\cos\phi)/(k+1),$$
$$y = a(\sin k\phi + k\sin\phi)/(k+1).$$

For each value of ϕ, (x, y) is the point of contact of the line $F=0$ with its envelope, which, being the locus of (x, y), is the well known epicycloid. Finally, (x, y) divides UV in the ratio $1:k$.

Also solved by F. C. Hall.

The Sum of p Consecutive Superfactorials

E 654 [1945, 42]. *Proposed by D. H. Browne, Buffalo, N. Y.*

Let $n!_1$ denote the coefficient of $x^n/n!$ in the expansion of $e^x/(1-x)$. Show that, for a prime p and any nonnegative integer k,

$$\sum_{n=k}^{k+p-1} n!_1 \equiv -1 \pmod{p}.$$

(*Cf.* E 488 [1942, 478].)

Solution by N. J. Fine, Lukas-Harold Laboratory, Indianapolis. Since $n!_1 = 1 + n(n-1)!_1$, we have

$$(n+p)!_1 = 1 + (n+p)(n+p-1)!_1 \equiv 1 + n(n+p-1)!_1 \pmod{p}.$$

Subtracting,

$$(n+p)!_1 - n!_1 \equiv n\{(n+p-1)!_1 - (n-1)!_1\}.$$

But $p!_1 - 0!_1 = 1 + p(p-1)!_1 - 1 \equiv 0$. Therefore $(n+p)!_1 \equiv n!_1$ for every n. Consequently

$$\sum_{n=k}^{k+p-1} n!_1 \equiv \sum_{n=0}^{p-1} n!_1 \pmod{p}.$$

Now, since

$$n!_1 = n! \sum_{k=0}^{n} \frac{1}{k!} = \sum_{k=0}^{n} \binom{n}{k} k!,$$

$$\sum_{n=0}^{p-1} n!_1 = \sum_{n=0}^{p-1} \sum_{k=0}^{n} \binom{n}{k} k! = \sum_{k=0}^{p-1} k! \sum_{n=0}^{p-1} \binom{n}{k}.$$

The inner sum is $\binom{p}{k+1}$; for it is the coefficient of x^k in the expansion of

$$\sum_{n=0}^{p-1} (1+x)^n = \frac{(1+x)^p - 1}{x} = \sum_{k=0}^{p-1} \binom{p}{k+1} x^k.$$

Since $\binom{p}{k+1} \equiv 0 \pmod{p}$ unless $k = p-1$, Wilson's Theorem gives

$$\sum_{n=0}^{p-1} n! \equiv (p-1)! \equiv -1.$$

Also solved by D. W. Alling, H. W. Becker, J. B. Kelly, John Riordan, E. P. Starke, and the proposer.

Twenty Terms of a Geometrical Progression

E 655 [1945, 42]. *Proposed by W. C. Rufus, Observatory of the University of Michigan*

The first term of a geometrical progression is a three-digit number, and the ratio is 9. Each term is divisible by the sum of its digits. Find the sequence having the largest number of terms. (The one beginning with 100 has thirteen terms.)

Solution by the Proposer. The sum of the digits of a product when 9 is the multiplier is 9 or a multiple of 9. Each term of the progression after the first, therefore, will contain powers of the prime factor 3. For an optimum progression the first term must contain the prime factors 2, 5, 7; *i.e.*, it must be a three-digit multiple of 70, divisible by the sum of its digits. Such numbers are 140, 210, 280, 420, 630, 700, 770, 840, 910. In those cases where the factor 2 occurs to the first power, the progression will break down for a term with digit-sum 36; so these may be omitted from consideration. Only five optimum first terms remain: 140, 280, 420, 700, 840. Of these 140 gives a sequence with 20 terms, the largest number found.

Thus the answer is

140, 1260, 11340, · · · , 189119240474218892460.

Solved incorrectly by Murray Barbour, D. H. Browne, J. E. Hanson, Robert Hoskins, Irving Kaplansky, and E. D. Schell, who all found the first term 280 or 840, and consequently only 18 terms altogether.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis 5, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known text-books or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4166. *Proposed by J. H. Butchart, Grinnell College.*

Given the straight lines l, l' and a point F. A variable circle through F and the intersection of l, l' cuts l and l' in A and A' respectively. The circles through F and tangent to l, l' at A, A' respectively meet again on a parabola tangent to l and l' and having F as focus.

4167. *Proposed by R. A. Staal, Student, University of Toronto*

What curves are self-reciprocal with respect to the conic $x^2+y^2=z^2$ (or the circle $x^2+y^2=1$)?

4168. *Proposed by Henry Scheffé, Princeton University*

The matrix $X=(x_{ij})$ is real $m\times n$ with $m\leqq n$. Let the ith row sum be $y_i=\sum_j x_{ij}$. The elements x_{ij} may vary subject to the conditions (1) $XX'=I$, (2) $y_1=y_2=\cdots=y_m$. Write $y_i=y$. Show that the maximum and minimum values of y are $\sqrt{n/m}$ and $-\sqrt{n/m}$ respectively.

4169. *Proposed by Victor Thébault, Tennie, Sarthe, France*

The tangents at the vertices A, B, C of a given triangle to its Feuerbach hyperbola form a triangle whose conjugate circle is tangent to the inscribed circle of ABC at its Feuerbach point.

4170. *Proposed by Victor Thébault, Tennie, Sarthe, France*

The powers of the vertices A, B, C of a given triangle with respect to the circles (ω_1), (ω_2), (ω_3) are respectively (ka^2, kb^2, kc^2), (kb^2, kc^2, ka^2), (kc^2, ka^2, kb^2), where a, b, c are the lengths of the sides of ABC. Find the loci of the centers ω_i as k varies. (2) The circumcenter of ABC is the centroid of triangle $\omega_1\omega_2\omega_3$ which remains similar to itself. (3) The straight line $\omega_2\omega_3$ is perpendicular to the join of the centroid and Lemoine point of ABC.

SOLUTIONS

Self Polar Triangle Inscribed in Another Conic

4063 [1942, 689]. *Proposed by H. S. M. Coxeter, University of Toronto*

In projective geometry the porism of triangles inscribed in one conic and self-polar for another is commonly proved by showing that if one such triangle exists,

we can find another with one vertex at *any* given point on the first conic. This statement is easily seen to be valid in complex geometry. Discuss its possible failure in real geometry.

II. *Solution by R. G. Stanton, University of Toronto.* Let the two conics be represented by homogeneous equations

$$yz + zx + xy = 0, \tag{1}$$

$$ax^2 + by^2 + cz^2 = 0. \tag{2}$$

The reference triangle is inscribed in (1), self-polar re (2). Let (x, y, z) be any point on (1). Its polar re (2) has the tangential coordinates $[ax, by, cz]$. Since the tangential equation for (1) is

$$X^2 + Y^2 + Z^2 - 2YZ - 2ZX - 2XY = 0,$$

this polar meets (1) in two points which coincide if and only if

$$a^2x^2 + b^2y^2 + c^2z^2 - 2bcyz - 2cazx - 2abxy = 0. \tag{3}$$

The four points of intersection of the two conics (1) and (3) give four points on (1) whose polars re (2) touch (1).

If a point P is on (1), let its polar QR, re (2), meet (1) in Q and R to form the self-polar triangle PQR. The problem is to determine for what positions of P this self-polar triangle is real. The transition from real to imaginary positions occurs when QR changes from a secant to an exterior line; thus the transitional stage is one in which QR is tangent to (1), and can occur for only four positions of P, corresponding to the roots of the quartic equation for x/y formed by eliminating z between (1) and (3). Three situations may arise:

(i) the quartic has four real roots;

(ii) the quartic has only two real roots;

(iii) the quartic has no real roots.

In case (i), the four real points will divide the conic (1) into four arcs, on two of which P may lie for real self-polar triangles. In case (ii) there will be only one such arc. In case (iii) every position on the conic will give a real triangle. (One such being already given, all must be real.) Intermediate situations, corresponding to coincident roots, occur when (1) and (2) are in contact.

Examples of these situations are readily provided. If $a=b=1$, we have two arcs of admissible positions for P, or only one, according as $c<-8$ or $-8\leqq c<0$. An instance of case (iii) occurs when (2) is

$$\tfrac{1}{4}x^2 + y^2 - z^2 = 0.$$

Editorial Note. A partial solution is given 1944, 171.

4098 [1945, 49]. An Alternant Type of Determinant

II. *Note by Arnold Dresden, Swarthmore College.* The result obtained in the solution [1945, 49] appears to be practically identical with Theorem 2 in the

article *On the generalized Vandermonde determinant and symmetric functions*, in the Bull. Am. Math. Soc., 1933, vol. 39, no. 6, p. 447.

Hyperboloid and Parallelepiped

4112 [1944, 167]. *Proposed by N. A. Court, University of Oklahoma*

If three rulers, chosen arbitrarily, of the same system of a given hyperboloid are taken for the edges of a parallelepiped, the diagonals of the parallelepiped meet in a fixed point.

Solution by Robert Steinberg, University of Toronto. Let A, B, C, D, A', B', C', D' be the vertices of the parallelepiped so that $ABCD$ and $A'B'C'D'$ are a pair of opposite faces joined by the edges AA', BB', CC', DD', let L, M, N be the points at infinity on AB, AD, AA' respectively, and let O be the point of intersection of the diagonals of the parallelepiped. Suppose that the three given generators are AA', BC, and $C'D'$. Then, three generators of the other system of the hyperboloid are CC', $A'D'$ and AB since each of these three lines meets each of the three given generators. Hence through L we have the two generators ABL and $C'D'L$. Thus the plane LBC' is the polar plane of L. Since AC' and BD' meet at O, this means that O lies in the polar plane of L. Hence, L lies in the polar plane of O. Similarly M and N lie in the polar plane of O, and hence O is the pole of the plane at infinity, *i.e.*, O is the center of the hyperboloid, a fixed point.

Solved also by the proposer, who referred to *Educational Times, Reprints*, vol. 3, 1903, p. 120, Q. 5778, Wilkins-Nanson.

Factorial Coefficients

4118. *Corrected. Proposed by Otto Dunkel, Washington University*

Show that

$$\sum_{t=0}^{n} (-1)^{n+t} \frac{t^{n+4}}{t!(n-t)!} = \frac{(n+4)(n+3)\cdots n}{6!8} [15n^3 + 30n^2 + 5n - 2], \quad n \geqq 0,$$

and that each member of this equality is a non-negative integer. If n is a negative integer, the right member is an integer; what meaning may be given to the result in this case?

Solution by J. S. Frame, Michigan State College. We expand the function $f(x) \equiv x^{-n}(1-e^{-x})^n$ in a power series $\sum f_r x^r$ in two distinct ways, and obtain two expressions for the number $N = (n+4)(n+3)(n+2)(n+1)f_4$, which are the left and right hand members respectively of the given identity.

First, if n is a positive integer, we have

$$f(x) = x^{-n} \sum_{t=0}^{n} (-1)^t \binom{n}{t} e^{-tx} = x^{-n} \sum_{t=0}^{n} \sum_{s=0}^{\infty} (-1)^{t+s} \binom{n}{t} \frac{t^s x^s}{s!}.$$

The coefficient f_4 of x^4 is obtained by setting $s = n+4$ in the final summation; from this we get N in the form of the left member:

$$f_4 = \sum_{t=0}^{n} (-1)^{t+n+4}\binom{n}{t}\frac{t^{n+4}}{(n+4)!}; \qquad N = \sum_{t=0}^{n} (-1)^{t+n}\frac{t^{n+4}}{t!(n-t)!}.$$

Secondly, we expand the nth power of the function $g(x) = (1-e^{-x})/x = \sum g_j x^j$ by means of the following lemma.

LEMMA. *If* $\sum_{r=0}^{\infty} f_r x^r = (\sum_{j=0}^{\infty} g_j x^j)^n$, *and* $f_0 = g_0 = 1$, *then*

$$f_m = \sum_{k=1}^{m}\binom{n}{k}\sum_{j} g_{j_1}g_{j_2}\cdots g_{j_k} \quad \text{where} \quad j_1 + j_2 + \cdots + j_k = m,\ j_s \geqq 1.$$

In this problem we have $g_j = (-1)^j/(j+1)!$,

$$f_4 = \binom{n}{1}g_4 + \binom{n}{2}(g_1g_3 + g_2g_2 + g_3g_1) + \binom{n}{3}(3g_1g_1g_2) + \binom{n}{4}g_1^4,$$

$$= \binom{n}{1}\frac{1}{5!} + \binom{n}{2}\left(\frac{1}{2!4!} + \frac{1}{3!3!} + \frac{1}{4!2!}\right) + \binom{n}{3}\frac{3}{2!2!3!} + \binom{n}{4}\frac{1}{16}.$$

$$N = \binom{n+4}{5} + \binom{n+4}{6}3\cdot4\cdot5\cdot6\left(\frac{1}{24} + \frac{1}{36}\right) + \binom{n+4}{7}\frac{4\cdot5\cdot6\cdot7}{8}$$

$$+ \binom{n+4}{8}\frac{5\cdot6\cdot7\cdot8}{16},$$

$$= \binom{n+4}{5} + 25\binom{n+4}{8} + 105\binom{n+4}{7} + 105\binom{n+4}{8}.$$

This expression defines an integer N for all integral values of n. It reduces at once to the right member of the given identity; since

$$N = \frac{(n+4)(n+3)\cdots n}{6!8}\,[6\cdot8 + 200(n-1) + 120(n-1)(n-2) + 15(n-1)(n-2)(n-3)].$$

When $n = -m$, $m \geqq 5$, the series for $f(x)$ must be replaced by

$$f(x) = x^m\sum_t (-1)^t\binom{-m}{t}e^{-tx} = x^m\sum_{t=0}^{\infty}\binom{m-1+t}{m-1}e^{-tx},$$

which converges for positive values of x. The number $N = (m-1)\cdots(m-4)f_4$ is

$$N = \binom{m-1}{4}\lim_{x\to0}\sum_{t=0}^{\infty}\frac{(m-1+t)!}{(m-1)!t!}\frac{d^4}{dx^4}(x^me^{-tx}), \qquad m = -n.$$

Solved by C. D. Olds, G. Pólya, and G. T. Williams.

Editorial Note. In the notation of 4108 [1945, 281] we have

$$(1)\qquad x^n = \sum_{r=1}^{n} {}_nQ_r x^{(r)}, \quad x^{(r)} = x(x-1)\cdots(x-r+1), \quad {}_{n+1}Q_r = {}_nQ_{r-1} + r\,{}_nQ_r,$$
$${}_nQ_1 = {}_nQ_n = 1, \quad {}_rQ_r = \Delta^r 0^n/r!.$$

Hence

$$(2)\qquad {}_{n+r}Q_n = \Delta^n 0^{n+r}/n! = (U-1)^n 0^{n+r}/n! = \sum_{t=0}^{n} {}_nC_t U^t (-1)^{n-t} 0^{n+r}/n!$$
$$= \sum_{t=0}^{n} (-1)^{n+t} t^{n+r}/t!(n-t)!.$$

It was shown also that ${}_{n+r}Q_n = \sigma_r(-n)$, where $\sigma_r(n) = {}_nP_{n-r}$ is the rth elementary symmetry function of $1, 2, \cdots, n-1$. In 3940 [1941, 641] and the above reference it is shown that

$$(3)\qquad {}_nP_{n-r} = \sigma_r(n) = n^{(r+1)} \sum_{t=1}^{r} (-1)^{r+t}\, {}^r a_t (n+t-1)^{(t-1)};$$

and formulas are given for the computation of a_t^r with the numerical results for $0 \leqq r \leqq 5$. Thus

$$(4)\qquad \Delta^n 0^{n+r}/n! = \sum_{t=0}^{n} (-1)^{n+t} t^{n+r}/t!(n-t)! = (n+r)^{(r+1)} \sum_{t=0}^{r} {}^r a_t (n-1)^{(t-1)}.$$

These formulas give the results desired in the problem. It is shown that the polynomial $\sigma_r(n)$ of degree $2r$ in n has the factor $n^2(n-1)^2$ if r is odd $\geqq 3$. Thus

$$(5)\qquad \sum_{t=0}^{n} (-1)^{n+t} \frac{t^{n+5}}{t!(n-t)!} = (n+5)^{(6)}(n+1)n(3n^2+7n-2)/2^5 5!3.$$

All the solvers replaced the erroneous $15n$ by $5n$. Olds obtained the results in (1) and (2) in a different notation, and then gave a table of his computations of ${}_{n+r}Q_n$ for $1 \leqq r \leqq 4$ which resulted in the equation of the problem. This was followed by developments of ${}_nP_r$ and ${}_nQ_r$ similar to those given by Weisner in the solution of 4108, but in a different form using in one case the generating function $(e^x-1)^n$. The solutions by Pólya and Williams also used the same generating function.

Gergonne Point

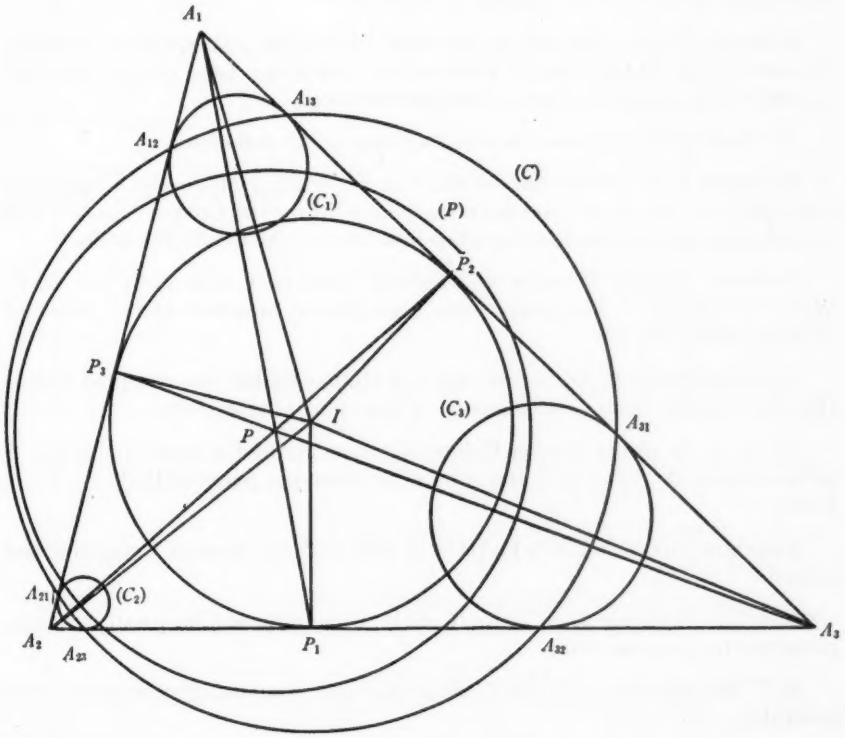
4119 [1944, 234]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

The straight lines joining the vertices of a triangle to the points of contact of the inscribed circle with the respective opposite sides meet in a point P. Show that the six points of contact of circles tangent to two sides and orthogonal to a given circle with center P are on a circle concentric with the inscribed circle.

Solution by Howard Eves, Syracuse University. Let $A_1A_2A_3$ be the given triangle, and let P_1, P_2, P_3 be the feet of the perpendiculars from the incenter I on the sides of the triangle, P_i lying opposite A_i. Designate the given circle with center P by (P) and let (C_1) be a circle orthogonal to (P) and touching A_1A_3 at A_{13} and A_1A_2 at A_{12}.

Draw circle (C) concentric with the incircle and passing through A_{13} and A_{12}. Let (C) cut A_1A_3 again in A_{31} and A_1A_2 again in A_{21} and A_2A_3 in A_{23} and A_{32}. Draw (C_2) touching A_2A_1 and A_2A_3 at A_{21} and A_{23}, and (C_3) touching A_3A_1 and A_3A_2 at A_{31} and A_{32}. Since I is the center of (C) it follows that all the segments P_iA_{jk} $(i\neq j\neq k\neq i)$ are equal. Hence P_i and A_i lie on the radical axis of C_j and C_k. Thus P is the radical center of the three circles (C_i), and since (P) is orthogonal to (C_1) it is also orthogonal to (C_2) and (C_3).

This is sufficient to establish the theorem.



Note. In connection with this problem see E 457 (vol. 48, no. 9, p. 636, Nov. 1941).

Editorial Note. The proposer remarked as follows: It is easy to show that the radical center of three circles (C_1), (C_2), (C_3) each tangent to two sides of a triangle, so that the points of contact are six points of a single circle, is the Gergonne point P in whatever manner the three tangent circles are chosen with their centers C_i on the internal bisectors. The theorem of the problem states the converse of this property.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to B. W. Jones, White Hall, Cornell University, Ithaca, New York.

Associate Professor R. S. Burington of Case School of Applied Science has been awarded the Meritorious Civilian Award from the U. S. Navy for his work as director and organizer of a research analysis group in the Research and Development Division of the Bureau of Ordnance.

Professor Garrett Birkhoff of Harvard University and Professor Norman Levinson of the Massachusetts Institute of Technology have been elected fellows of the American Academy of Arts and Sciences.

Dr. Paul Erdös has been awarded a Guggenheim fellowship.

Reverend J. T. O'Callahan of the College of the Holy Cross, Lieutenant Commander in the Navy, has been recommended for the Congressional Medal of Honor for outstanding bravery while in service on the U.S.S. *Franklin*.

Professor Hassler Whitney of Harvard University and Professor E. P. Wigner of Princeton University have been elected members of the National Academy of Sciences.

Associate Professor Harriet W. Allen of Hollins College has accepted a position with the Air Reduction Company of Stamford, Connecticut.

Dr. L. A. Aroian of Hunter College has been granted a leave of absence to serve as research associate in the applied mathematics panel at Berkeley, California.

Associate Professor Edith I. Atkin of Illinois State Normal University has retired.

Associate Professor I. A. Barnett of the University of Cincinnati has been promoted to a professorship.

A. F. Bartholomay of Keuka College has been promoted to an assistant professorship.

Associate Professor E. F. Beckenbach of the University of Texas has been appointed to an associate professorship at the University of California at Los Angeles.

Assistant Professor Z. W. Birnbaum of the University of Washington has been promoted to an associate professorship.

Assistant Professors I. W. Burr and H. F. S. Jonah of Purdue University have been promoted to associate professorships.

Assistant Professor Herbert Busemann of the Illinois Institute of Technology has been appointed to an assistant professorship at Smith College.

Professor Theodosia T. Callaway of Stephens College, Columbia, Missouri, has retired.

Associate Professor R. H. Cameron of the Massachusetts Institute of Technology has been appointed to a professorship at the University of Minnesota.

Professor L. M. Coffin of Coe College, Cedar Rapids, Iowa, has retired with the title professor emeritus.

Professor A. R. Congdon of the University of Nebraska has retired.

Dr. Helen W. Dodson has been appointed to an associate professorship at Goucher College.

Professor Albert Einstein of the Institute for Advanced Study has retired with the title professor emeritus.

Norman Gunderson of Cornell University is now serving in the Allegany Ballistics Laboratory, Cumberland, Maryland.

C. F. Hall of Rensselaer Polytechnic Institute has been promoted to an assistant professorship.

Dr. Margaret M. Hansman of Colorado College has been promoted to an assistant professorship.

Assistant Professor M. L. Hartung of the University of Chicago has been appointed associate professor of the teaching of mathematics.

Dr. F. E. Hohn of the University of Arizona has been appointed to an assistant professorship at Guilford College, Guilford College, North Carolina.

Assistant Professor H. T. Karner of Louisiana State University has been appointed associate professor of mathematics and dean of men.

Dr. Anne L. Lewis of the University of Chicago has been appointed to an assistant professorship at the Woman's College of the University of North Carolina.

Dr. Murray Mannos of the University of Notre Dame has been promoted to an assistant professorship.

Dr. Helene Reschovsky of Russell Sage College has been promoted to an assistant professorship.

Professor J. B. Reynolds of Lehigh University has been appointed head of the department of mathematics and astronomy.

Professor J. B. Rosenbach of the Carnegie Institute of Technology has been appointed acting head of the department of mathematics.

Professor N. E. Rutt of Louisiana State University has been given leave of absence to teach mathematics in the United States Army Study Center abroad.

Assistant Professor Henry Scheffé of Syracuse University has been granted leave of absence to serve as senior mathematician with the Princeton University station of division 2 of N.D.R.C.

Professor E. R. Smith has retired from the chairmanship of the department of mathematics at Iowa State College. Professor D. L. Holl will succeed him as chairman.

Associate Professor P. A. Smith of Columbia University has been promoted to a professorship. He will serve as executive officer of the department of mathematics for a term of three years.

Dr. R. H. Sorgenfrey of the University of California at Los Angeles has been promoted to an assistant professorship.

Associate Professor Marion E. Stark of Wellesley College has been promoted to a professorship. Assistant Professor Helen G. Russell has been promoted to an associate professorship.

Professor R. P. Stephens of the University of Georgia has retired.

Professor J. L. Synge has returned to the Ohio State University after service overseas.

Professor C. F. Thomas of Case School of Applied Science has retired.

Dr. A. R. Turquette of Cornell University has been appointed visiting assistant professor of philosophy at the University of Illinois.

The following appointments to instructorships have been announced:

East Carolina Teachers College, Greenville, North Carolina: Dr. Ethel Sutherland.

Rutgers University: Edmund Churchill, A. G. Makarov, Dr. L. M. Rauch.

President Emeritus W. A. Bratton of Whitman College died in June 1945. He was a charter member of the Association.

Professor P. J. da Cunha of the University of Lisbon died on February 4, 1945.

Professor K. S. K. Iyengar of the University of Mysore, Bangalore, India, died June 23, 1944.

Professor S. T. Sanders, Jr., of Southwestern Louisiana Institute died on April 10, 1945.

GENERAL INFORMATION

EDITED BY C. V. NEWSOM

Send information of especial interest to mathematicians, exclusive of personal items, to C. V. Newsom, Oberlin College, Oberlin, Ohio.

THE CRISIS IN TRAINING FOR THE SCIENTIFIC PROFESSIONS

In recent months, many scientists have forcefully pointed out that present manpower policies in the United States have seriously endangered the postwar supply of scientists. Especial reference may be made to two articles written by Dr. M. H. Trytten, Director of the Office of Scientific Personnel of the National Research Council; these appeared in *Science* (101: 172–173, 1945) and in *The Scientific Monthly* (LX: 37–47, 1945). On May 28, 1945, the American Council on Education and the Office of Scientific Personnel jointly issued a special bulletin of 32 pages asking for the resumption of training for the scientific professions. Because of the critical nature of the facts contained in this bulletin, a digest of it follows.

War has increased the need of the nation for technical and professionally trained manpower; yet, at the same time, it has decreased and nearly stopped the flow of able-bodied men into these fields so essential to the national health, safety, and interest. War has, also, brought a tremendous increase in our dependence upon specialized knowledge and skill but has sharply curtailed the training of selected individuals for service in these fields.

As America looks to the reconversion period, the increasing demand and greater dependency upon scientific and technological developments will increase. If these demands are to be met, if America is to have essential security in terms of national health, if we are to meet successfully the competition of nations that have preserved their professional personnel, we must begin now to take such steps as may partially make up for the growing deficit caused by the reduced flow of men into these fields. Either by administrative action or by legislation, we must reverse the policy established over the past eighteen months.

A conservative estimate indicates that even if enrollments in medical schools are maintained at present levels, there will result a shortage of some 19,000 doctors available for civilians in the postwar period as compared with the prewar period. Despite this fact, government policies and regulations now in effect will inevitably result in either (1) a sharp reduction in medical school freshmen enrolled in late 1945 and especially in 1946, or (2) the selection of large numbers of poorly qualified applicants. To insure an adequate supply of physicians in the future, about 8,000 students graduating from high school in 1945 should be selected to enter college on a premedical program and deferred from active military service. Selection should be based on high school performance, aptitude in

science as determined by special tests, motivation toward science and medicine, intelligence and emotional stability.

In considering the postwar supply of dental manpower upon which the people of the United States must depend for adequate oral health care, it is desirable to point out that there was a real shortage of dentists in the United States in the years immediately preceding the outbreak of World War II. Yet, under present conditions, the dental schools of the country will be unable to secure a freshman class in 1945. The dental A.S.T.P. has been discontinued; the Navy V-12 program lacks sufficient candidates to provide more than four per cent of the total enrollment capacity of the dental schools; the availability of discharged veterans has proved disappointing; 4 F's and women make up only about eight per cent of the total current enrollment in dental schools. Instead of the maximum possible enrollment of 3,000 freshman students in the thirty-eight dental schools that is imperative in present circumstances, there will be an actual enrollment of not more than 360. The oral health situation which now confronts the people is critical and unless there is prompt action to relieve this condition through pending legislation or a reversal of present policies regarding deferment, the situation of the nation is jeopardized.

The extent to which industry and the life of the country generally are dependent on highly trained personnel is so very great and the attendant change in American industrial and commercial processes has been so rapid that full appreciation of the situation is rare among the people and in government. The absorption of scientists and engineers has been at a very rapid rate as American industry has steadily exploited the fruits of technological advances, and processes have become more exact and more intricate. It is this alertness to scientific advance which has given our industry its enormous productivity power and has made it possible for America to commence with a handicap and yet outstrip the world in equipping its fighting arms. The war has not stopped nor reversed this technological trend. In some directions technological development has been stimulated by the war, in others it has been slowed down. For complete recovery the trend must be recaptured in all fields.

The latest government sponsored survey seeking to measure the demand in industry for highly trained engineers and scientists was made a year ago. The National Roster of Scientific and Specialized Personnel published in May, 1944, a report based on a survey made through the facilities of the U.S.E.S. The National Roster estimated at that time that the accumulated total need on the part of the American industry was likely to be 40,000 scientists and engineers by June, 1944, about the number graduated in eighteen months in American colleges and universities in normal times. This means that industry was already a year and a half behind at that time in recruiting its forces to meet new demands and to recoup losses to Selective Service. It is now one more year behind.

Recently Dean A. A. Potter of Purdue University conducted a poll of something over one hundred companies to determine the present situation with regard to highly trained personnel. His letter of inquiry asked for the number

of engineers needed by these companies to meet critical needs at the present time. The results show that about 48 engineers per company are needed by these firms on the average to handle adequately their present urgent war responsibilities. This is a shortage of nearly 5,000 engineers for this small cross section of American industry alone. These companies estimated that to meet their reconversion needs an additional thirty-six engineers per company would be needed. To appreciate the import of these figures it must be borne in mind that reconversion is largely dependent on highly skilled personnel during the periods of research, development, and plant engineering. Mass employment can not occur until these preparatory steps have been completed.

It must not be overlooked that in the future foreign nations will contribute to the demand for American technological personnel. Technologically trained brains have become a commodity of supreme importance to military and economic well-being. Foreign countries have not failed to appreciate this.

In spite of this great potential demand for engineers, the Society for the Promotion of Engineering Education reports enrollments in engineering in American colleges of something over 110,000 in 1940–41, the last "normal" year, about 50,000 in 1943–44 and about 38,000 in 1944–45; of this latter group, however, 17,000 were freshmen, most of whom are awaiting induction.

The field of physics is of especial interest because of its particular and strategic importance in this war and in all research for military purposes, as well as for its importance in the research work underlying recovery and reconversion. At all levels of training, the reduction in the training of physicists has been drastic. The number of persons achieving the doctorate level has dropped each year since 1941, the drop each year being of the order of twenty-five per cent of the 1940 number of graduates. Since this number includes many who have been able to complete their training in spite of being heavily occupied with war research, the picture fails to tell the whole story. It is also clear that the flow of persons trained to the doctorate level can not be resumed in the future until the lower levels have regained their rate of flow. Thus, not only will there be a shortage of physical scientists at all levels but the deficit must be projected into the future to perceive the cumulative effect on our scientific competence.

In the case of chemistry, the figures reported by the American Chemical Society show enrollments of male students following the same pattern. Enrollment is made up almost entirely of women and a small per cent of 4 F's. Since women, on the average, do not remain long in the professions and less frequently go on to achieve higher levels of training, the contribution to the scientific competence of the nation is not proportionate to the numbers enrolled. Dr. Charles Parsons, of the American Chemical Society, states that at the present time the need for highly trained chemists is many times the available supply and is growing progressively even more serious.

In the case of mathematics, exact data are not available but every indication is that the situation is probably worse than in other fields on account of more adverse Selective Service policies.

Scientists trained to the doctoral level are the future leaders of American science and are the staff and line officers of the scientific campaigns of the future, be they in civilian or military researches. It is amazing that in the very period when the worth of such personnel is so brilliantly demonstrated, the record shows we have canceled our production of reserves. There is no great possibility that the flow of trained personnel can appreciably pick up before the end of resistance in the Pacific. Selective Service quotas do not show signs of being reduced and if they are, the reduction will only affect the induction of older persons. It seems realistic to look for no large scale resumption of training in the technological branches in American schools before 1947. It will thus be about 1950 before industry can expect to begin large scale recruiting again and then only at the baccalaureate level. Full scale resumption of training at the doctorate level must occur much later.

It must be emphasized that the stoppage of training is much more serious than figures indicate. Figures can not give a measure of the effects produced which will be difficult to counteract, such as dislocation and dispersal of faculty personnel, lack of trained faculty replacement and accelerated attrition in the ranks of scientists.

What information is available on Russia indicates that training in engineering and the sciences has been vastly stimulated during the war. Edgar Snow has reported in books and articles that much emphasis has been placed on education at all levels and in the technological branches particularly. Russian scientists in the United States report that the training facilities of the Soviet Union are in full utilization. In England there has been throughout the war a very intelligent attitude toward training in the technological branches. In these branches a student who is in good standing and who is deemed by his faculty as likely to finish his training in the required time is held at the university to finish his course. At each university there is an office known as a Joint Recruiting Board which represents the university and the services. If the faculty recommends to this board the reservation of the student so he may finish his course, the board reserves him. They follow his progress and on graduation advise with him so as to indicate to him where he can best be used in the national interest either in the services or in industry. As a result of this policy England has had a steady flow of students into the critical categories, at least equal to the prewar norm, and probably about fifty per cent above it in physics and some branches of engineering. In the case of Canada a substantially similar course of action has been taken; approximately the same method of control as in England has been adopted and stimulation of training in the technological branches has resulted. A news release in the *New York Times* (February 28, 1945) entitled "Facts and Figures of Australia at War" reports that at the University of Sydney in 1943 medical enrollments were up eleven per cent, sciences forty-three per cent and engineering fifty per cent. It is thus clear that among our allies the training of students in the scientific branches is at least equal to that of pre-war years and in some branches definitely above pre-war levels.

Recognizing the seriousness of the situation in the scientific professions, conferences have been held with both Selective Service and the Inter-Agency Committee of the War Manpower Commission. While both of these agencies frankly recognized the serious situation that would inevitably result in stopping the flow of men into professions essential to the national interest, they stated that it was inadvisable to reestablish student deferment by regulation. They pointed out that the quota demands of the Army and Navy were such that not only could there be no deferment for 18 year old men but that increasing pressure would be brought to bear upon men up through age 33. The present policy of Selective Service is clearly stated in a letter dated April 27, 1945, from which the following paragraph is quoted:

"The urgent manpower requirements of the armed forces for young men who are physically and mentally qualified to engage in active combat service necessitate an increasingly strict examination of registrants, ages 18 through 29, for purposes of occupational classification. To the extent that registrants in this age group are deferred on occupational grounds local boards find it necessary to induct registrants over 30 years of age in order to meet the calls of the armed forces. The manpower requirements of the armed forces plus the manpower needs in war production do not permit deferment of students who wish to undergo training in specialized fields except as provision has been made for students in medicine, dentistry, veterinary medicine, or osteopathy."

Even this brief summary of the increasing need and the decreasing supply to meet the need in the scientific professions indicates that action is imperative. There are two possible courses of action—modification of existing administrative policies or legislation. The Selective Service and Training Act of 1940 and supplementary regulations which have the full effect of law are adequate to authorize the Selective Service System to defer such numbers as are necessary to meet the imperative need in essential fields. However, if this need is to be met through administrative action it will entail a complete reversal of the policy which began with the curtailment of student deferment in 1942. That Selective Service does not now contemplate such a reversal policy is indicated in the quotation just above. If such reversal of policy is not made through existing administrative agencies, it entails the necessity of immediate legislative action. Bills providing for student deferment on a quota basis and in terms only of demonstrated needs have already been introduced into the 79th Congress.

An editorial in the May 1 issue of *Industrial and Engineering Chemistry* entitled "Shall We Sell America Short Scientifically?" concludes with the following:

"With reckless abandon, our Government is selling America technologically short. Unless an aroused public demands an immediate investigation and appropriate action, our country is headed for a second- and third-rate role in the postwar period. God help us if we become engaged in a third World War in the next 25 years."

Local Board Memorandum 115 of the Selective Service System, as amended June 22, 1945, indicates no change in policy in the induction of young men ages 18 through 29. In fact, Section 4 of this Memorandum states that "certifying agencies (The National Roster has been the certifying agency for mathematics since March 21, 1945) are being required to reduce the total number of outstanding certifications" for deferment from military service.

In view of such trends in Selective Service policy, the Executive Committee of the A.A.A.S. at a meeting held on April 22 voted to endorse and urge the passage by Congress of Bill H.R. 2827. This Bill, known as the McDonough Bill, was introduced in the House of Representatives on April 2, 1945, and is quoted below.

"Be it enacted by the Senate and House of Representatives of the United States of America in Congress assembled, That (1) in order to make possible the training, education, and availability of such numbers of persons as are necessary for the health, safety, and welfare of the Nation in the sciences of fundamental importance to the conclusion of the war, the safe reconversion of the national life to the ways of peace, and the Nation's potency in the world's future economy, the President shall, under such rules and regulations as he may prescribe, provide for the deferment from military service of

(a) not to exceed 20,000 young men annually for training to meet essential needs in the physical sciences and in their application to technology and engineering, and of teachers to conduct said training program; and

(b) 15,000 trained scientists and engineers now employed in research or by industry in work essential to the health, safety, and welfare of the Nation.

Section 2. "The discharge or assignment to essential civilian pursuits of

(a) not to exceed 20,000 technically-trained enlisted men, especially chemists, chemical engineers, physicists, and mathematicians, not utilizing their highest skills in the practice of their professions in the Army or Navy, to industries and educational institutions urgently in need of such men; and

(b) not to exceed 15,000 enlisted men partially trained in those branches of science and engineering in which shortages exist, but whose collegiate training was interrupted by military service and who had shown promise of completing with distinction their preparation for professional work; providing they undertake to immediately resume and continue their collegiate training to graduation.

Section 3. "No provision of law in force on the date of enactment of this Act shall be construed to authorize any action inconsistent with the provisions and purposes of this Act."

Imaginaries. The danger, however, is not less real because it is imaginary; imagination acts upon man as really as does gravitation and may kill him as certainly as does a dose of prussic acid.—Frazer, *Golden Bough*, p. 223 of the condensed edition, published by Macmillan.—*Arnold Dresden.*

THE MATHEMATICAL ASSOCIATION OF AMERICA

THE MARCH MEETING OF THE SOUTHERN CALIFORNIA SECTION

The twenty-fifth regular meeting of the Southern California Section of the Mathematical Association of America was held at the George Pepperdine College in Los Angeles on Saturday, March 10, 1945. Professor P. G. Hoel, Chairman of the Section, presided at the morning and afternoon sessions.

The attendance was sixty-four, including the following thirty-nine members of the Association: O. W. Albert, Clifford Bell, E. T. Bell, L. T. Black, W. D. Cairns, Myrtie Collier, P. H. Daus, D. C. Duncan, W. H. Glenn, Jr., P. C. Hammer, E. J. Hills, P. G. Hoel, D. H. Hyers, C. G. Jaeger, Glenn James, E. M. Justin, G. R. Kaelin, L. C. Lay, Margaret B. Lehman, Ada A. McClellan, G. F. McEwen, P. M. Niersbach, W. B. Orange, W. T. Puckett, Jr., H. R. Pyle, V. V. Quilliam, W. C. Randals, E. C. Rex, J. M. Robb, G. E. F. Sherwood, R. H. Sorgenfrey, D. V. Steed, V. C. Throckmorton, S. E. Urner, F. A. Valentine, Morgan Ward, R. L. White, Euphemia R. Worthington, Max Zorn.

At the business meeting the following officers were elected for the next year: Chairman, R. P. Dilworth, California Institute of Technology; Vice-Chairman, H. J. Hamilton, Pomona College; Program Committee, P. C. Hammer, E. J. Hills, D. H. Hyers, and P. H. Daus, Secretary. It was decided to hold the next meeting at the California Institute of Technology in Pasadena on March 9, 1946.

The following papers were presented.

1. *Mathematics needed for the training of nurses*, by Dr. E. J. Hills, Los Angeles City College.

It was stated that the mathematics needed by nurses includes decimal fractions, percentage, and proportions. Special devices for the solution of typical problems were explained. Illustrative problems pertaining to drugs and solutions were cited. The discussion was based upon the speaker's recent book entitled *Arithmetic of Drugs and Solutions*.

2. *Some fundamentals of freshman mathematics*, by Professor O. G. Harrold, Pomona College, introduced by Professor C. G. Jaeger.

This address consisted of a brief survey of some unusual efforts to re-examine matters of rigor in an introduction to mathematics at the college and pre-college level. The speaker described a course now being developed in which the emphasis is on form rather than content.

3. *The foundations of the theory of probability*, by Professor Hans Reichenbach, University of California at Los Angeles.

The speaker approached the foundations of the theory of probability from an axiomatic point of view, and discussed the philosophic implications when it is applied to physical situations.

4. *Heron's problem*, by Professor E. T. Bell, California Institute of Technology.

As generalized by Tannery, Heron's problem is the problem of solving the simultaneous system

$$a(x + y) = u + v, \qquad xy = buv$$

in which a and b are known integers, for integral values of x, y, u, and v. Several individuals have written papers which purported to present the complete solution of this problem, but in each case the work was incomplete, inexplicit, or unproved. The speaker presented a complete solution with its proof. By suitable specialization of the parameters in the complete solution, the special cases $a = b$ and $a = 1$ considered by Heron, Planude, and others can be solved.

5. *The use of conformal transformations to solve certain practical problems in aerodynamics*, by Dr. G. H. Peebles, Douglas Aircraft Company, introduced by Professor P. G. Hoel.

This paper dealt with the problem of designing airfoils, ducted airfoils, intakes, turbine blades, and slotted flaps to give prescribed flow patterns.

6. *Applications of mathematics—analysis by means of mass spectrometer*, by Miss Sibyl N. Rock, Consolidated Engineering Corporation, introduced by Professor G. E. F. Sherwood.

Miss Rock employed lantern slides to explain the mass spectrometer and its use in making certain chemical analyses. She also described an electric analogue computer used to solve a system of simultaneous linear equations arising in connection with the work. The time required for solution by means of this machine and by the usual calculator methods was stated. The question of errors was also considered.

7. *Educational problems in statistical quality control*, by Dr. P. C. Hammer, Lockheed Aircraft Corporation.

The speaker discussed the problem of presenting the quality control course to men with diverse educational backgrounds. He stated that at the Lockheed Aircraft Corporation the problem has been partially solved by presenting statistical concepts by means of experiments. Some of the materials used in these experiments were washers, rivets, playing cards, and ball bearings. The success of this type of instruction in statistics indicates that similar methods could be profitably employed in mathematics courses.

P. H. DAUS, *Secretary*

THE ANNUAL MEETING OF THE METROPOLITAN NEW YORK SECTION

The annual meeting of the Metropolitan New York Section of the Mathematical Association of America was held at the Polytechnic Institute of Brooklyn, Brooklyn, New York, on Saturday, April 21, 1945. Professor Jewell Hughes Bushey, Chairman of the Section, presided.

The attendance was one hundred and eight, including the following forty-seven members of the Association: Claire F. Adler, R. G. Archibald, Aaron Bakst, Brother Bernard Alfred (Welch), Frank Boehm, C. B. Boyer, Benjamin Braverman, A. B. Brown, Jewell Hughes Bushey, H. R. Cooley, T. F. Cope, W. H. H. Cowles, W. H. Fagerstrom, J. M. Feld, R. M. Foster, Marion C. Gray, Mary W. Gray, George Grossman, C. C. Grove, C. E. Heilman, L. S. Hill, Joseph Jablonower, Herman Karnow, Edna E. Kramer-Lassar, Helen L. Kutman, Nathan Lazar, C. H. Lehmann, Joseph Milkman, F. H. Miller, P. B. Norman, L. F. Ollmann, Max Peters, L. M. Reagan, Moses Richardson, John Riordan, S. G. Roth, Charles Salkind, Harry Schor, Aaron Shapiro, James Singer, F. E. Smith, E. R. Stabler, H. E. Wahlert, Etta A. Waite, Alan Wayne, D. E. Whitford, John Williamson.

At the business meeting the following officers were elected for the coming year: Chairman, F. H. Miller, Cooper Union; Vice-Chairman, H. E. Wahlert, New York University; Secretary, C. B. Boyer, Brooklyn College; Treasurer, Aaron Shapiro, Midwood High School.

The following program was presented:

1. *Demonstrative algebra*, by Professor E. R. Stabler, Hofstra College.

The speaker outlined a simple logical unit of algebraic postulates and theorems, essentially the elementary theory of number fields, for possible use in courses in high school or college algebra. Some advantages which might occur from the teaching of such a unit were cited.

2. *Triangular permutations*, by John Riordan, Bell Telephone Laboratories.

Permutations are called triangular when they are subject to a set of conditions on positions forbidden to elements which appear as a triangle in the square array formed with elements as columns and positions as rows. The properties and the problem of the enumeration of such permutations were discussed.

3. *Hadamard's determinant theorem*, by Professor John Williamson, Queens College.

Professor Williamson defined a Hadamard or H-matrix to be a square matrix of order $n>1$, each element of which is ± 1, and whose determinant has the maximum possible value $n^{n/2}$. He outlined the proof by which Paley showed that, if $n\equiv 0 \pmod 4$ and $n\leqq 200$, an H-matrix of order n exists except possibly for $n=92, 116, 156, 172, 184, 188$. He stated that it could be shown by other methods that an H-matrix of order 172 does exist, and that by generalization of Paley's methods the existence of H-matrices of certain other orders can be established. He also considered the maximum values Δ_n of the determinants of such matrices for certain small values of $n \not\equiv 0 \pmod 4$. These values of n were 3, 5, 6, and the corresponding values of Δ_n were 2^2, $3\cdot 2^4$, and $5\cdot 2^6$, respectively. It was remarked that $9\cdot 2^6 \leqq \Delta_7 \leqq 10\cdot 2^6$, and it seems probable that $\Delta_7 = 9\cdot 2^6$.

4. *Changing objectives in the teaching of algebra and trigonometry in the senior high schools*, by Benjamin Braverman, Seward Park High School.

In algebra and trigonometry as taught, say in 1915, the content was arranged topically and developed logically. Each topic was developed in all its theoretical ramifications, and highly involved techniques were taught. Now the content is organized in a psychological way, and selected because of its direct connection with the life situations of the student. The emphasis is on the understanding of concepts; skills and techniques are taught only as needed, and in problem situations that have meaning to the pupil.

5. *Changing objectives in the teaching of geometry in the senior high schools*, by Samuel Welkowitz, Franklin K. Lane High School, introduced by Dr. Nathan Lazar.

Mr. Welkowitz advocated that geometry be exhibited as an illustration of the scientific method, that the role of deduction and the nature of proof be emphasized, that we lift the pupil from the second to the third dimension, and that we provide practice on more useful constructions without restriction to the straightedge and compasses. He also recommended a modification of the order of topics to permit a gradual approach to formal deduction, the selection of the easier topics for the first semester, and the introduction of a greater amount of algebraic and arithmetic drill.

H. E. WAHLERT, *Secretary*

CALENDAR OF FUTURE MEETINGS

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN
ILLINOIS
INDIANA, Indianapolis, October 19, 1945
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA
NORTHERN CALIFORNIA, Berkeley, January 26, 1946
OHIO, April 4, 1946
OKLAHOMA
PHILADELPHIA, Philadelphia, December 1, 1945
ROCKY MOUNTAIN
SOUTHEASTERN
SOUTHERN CALIFORNIA, Pasadena, March 9, 1946
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE
WISCONSIN, Milwaukee, May, 1946

BOOKS FROM THE GINN LIST

★ **ROSENBACH-WHITMAN: COLLEGE ALGEBRA REVISED EDITION**

A clear, simple, rigorous presentation of the fundamentals of college algebra. Many illustrative examples showing the application of basic principles. An abundance of graded exercises. $2.60

★ **ROSENBACH-WHITMAN-MOSKOVITZ: PLANE AND SPHERICAL TRIGONOMETRY**

Outstandingly successful, this trigonometry is known for its clarity of exposition, soundness of method, and flexibility of arrangement. $2.30 With tables $2.50

★ **MASON-HAZARD—BRIEF ANALYTIC GEOMETRY**

A one-semester course. Simplicity is achieved by presenting work in the straight line before the general discussion of equation and locus. Plenty of drill exercises. $2.25

Prices subject to discount

Boston New York Chicago Atlanta Dallas Columbus San Francisco

GINN AND COMPANY

Successful College Mathematics Tests

W. L. Hart, W. A. Wilson, and J. I. Tracey's

● FIRST YEAR COLLEGE MATHEMATICS

Essentials of College Algebra (Hart); Plane and Spherical Trigonometry with Applications (Hart); Analytic Geometry (Wilson & Tracey). Tables. *886p. $4.00*

BRIEF EDITION (Hart)

Identical with the complete edition except for the omission of Analytic Geometry. *606p. $3.00*

A. L. Nelson, K. W. Folley, and W. M. Borgman's

● CALCULUS

Presented from the point of view of beginning students who need the subject primarily as a tool in engineering and other scientific fields. *366p. $2.75*

D. C. HEATH AND COMPANY

Boston New York Chicago Atlanta San Francisco Dallas London

New Edition

Outline of the History of Mathematics

by RAYMOND CLARE ARCHIBALD

Fifth edition, June 1941, ii, 76 pages

THIS thoroughly revised and considerably enlarged edition is published by the Mathematical Association to meet a constant demand for up-to-date information on the History of Mathematics, not available in any other single English work. The syllabus, twenty pages of references to other material and sources, and an entirely new Index of Names, provide an excellent basis for a teacher to conduct a semester or year course in this field, or for a student wishing greatly to extend his knowledge. Earlier editions of the Outline have been reviewed very favorably throughout the world. Our expectation is that this fifth edition published within ten years, and more fundamentally revised and extended than any other edition, will continue to meet a need in this country and elsewhere.

Price 75 cents a copy, postpaid, remittance with order

No discount in price to anyone

MATHEMATICAL ASSOCIATION OF AMERICA
McGRAW HALL, CORNELL UNIVERSITY
ITHACA, NEW YORK

The Rhind Mathematical Papyrus

De Luxe Edition

Volume I, Translation and Commentary
Volume II, Photographic Plates and Fac-Simile Reproduction

Members of the Association may procure copies at $20.00 per set through the office of the Secretary, McGRAW HALL, Cornell University, ITHACA, N.Y. Non-members order through the Open Court Publishing Company, La Salle, Ill., at $25.00 per set.

THIS IS A MAGNIFICENT WORK and should be in every college library. The edition is absolutely limited. Most of these sets are already sold, and no more will be available when this edition is exhausted.

ANNALS OF MATHEMATICS STUDIES

1. **Algebraic Theory of Numbers**
   By HERMANN WEYL 227 pp. $2.35
2. **Convergence and Uniformity in Topology**
   By JOHN W. TUKEY 95 *pp.* $1.50
3. **The Consistency of the Continuum Hypothesis**
   By KURT GÖDEL 68 *pp.* $1.25
4. **An Introduction to Linear Transformations in Hilbert Space**
   By F. J. MURRAY 135 *pp.* $1.75
5. **The Two-Valued Iterative Systems of Mathematical Logic**
   By EMIL L. POST 122 *pp.* $1.75
6. **The Calculi of Lambda-Conversion**
   By ALONZO CHURCH 77 *pp.* $1.25
7. **Finite Dimensional Vector Spaces**
   By PAUL R. HALMOS 201 *pp.* $2.35
8. **Metric Methods in Finsler Spaces and in the Foundations of Geometry**
   By HERBERT BUSEMANN 247 *pp.* $3.00
9. **Degree of Approximation by Polynomials in the Complex Domain**
   By W. E. SEWELL 246 *pp.* $3.00
10. **Topics in Topology**
    By SOLOMON LEFSCHETZ 139 *pp.* $2.00
11. **Introduction to Non-Linear Mechanics**
    By N. KRYLOFF and N. BOGOLIUBOFF 108 *pp.* $1.65
12. **Meromorphic Functions and Analytic Curves**
    By HERMANN WEYL 277 *pp.* $3.50
13. **Introduction to Mathematical Logic (Part 1)**
    By ALONZO CHURCH 119 *pp.* $1.75

The prices above are subject to an educational discount of fifteen per cent.

PRINCETON UNIVERSITY PRESS

Princeton, New Jersey

BOOK NEWS

PLANE TRIGONOMETRY

REVISED EDITION

By RAYMOND W. BRINK

THE distinctive method that has made outstanding successes of all of Brink's mathematics books characterizes this text, which is also notable for the clarity and conciseness of its style; for its emphasis on the practical uses of trigonometry; for its immediate application of principles to problems; for its great number of illustrative examples worked out in the text; for the flexibility of its organization, which makes the book adaptable to courses of various lengths and purposes; and for the great abundance of its exercise problems. 222 pages. With tables, $2.20. Without tables, $1.85.

D. APPLETON-CENTURY COMPANY

35 West 32nd Street New York 1, New York

CALCULUS

by Dr. Lyman M. Kells, Professor of Mathematics, U. S. Naval Academy

Dr. Kells has written a calculus distinguished for its outstandingly improved approach. It makes full use of review as an aid to learning. By studying and observing, the student becomes acquainted with a technique. Later he returns to it in a formal discussion with more complex situations.

Emphasis is placed on reasoning and understanding, rather than strictly formal calculus. The theoretical side of calculus has not been neglected; highly important concepts are considered early in the text. The author takes advantage of the student's initial curiosity to present him with ideas. He reintroduces the material when sufficient background has been developed for complete comprehension.

The explanations are consistently thorough, the problems varied and abundant.

511 pages *College List $3.75*

Adoptions

Princeton University
United States Naval Academy
Boston University
Harvard University
Johns Hopkins University
University of Chicago
University of Vermont
New York University
University of Southern California
Northwestern University
Brown University
City College of New York
University of Rochester
Massachusetts Institute of Technology
Fordham University
Brooklyn College
University of California at Los Angeles

and many more

MATHEMATICS OF FINANCE

Second Edition, Revised and Enlarged

by T. M. Simpson, Z. M. Pirenian, B. H. Crenshaw

• Divided into two parts, this text treats first Commercial Algebra and then presents an introduction to the mathematical theory of compound interest, annuities, life insurance, and statistics.

(Part II, separate . . . $2.50)

469 pages (with tables) *College List $3.75*

PLANE TRIGONOMETRY

by Fred W. Sparks and Paul K. Rees

• This text abandons the traditional order of topics so as to present the relationships between the apparently different trigonometric identities more effectively.

237 pages (with tables) *College List $2.00*
tables alone . . . 60¢ *without tables . . . $1.50*

for approval copies, address

PRENTICE-HALL, INC. 70 FIFTH AVENUE, NEW YORK 11

McGraw-Hill Books of Unusual Interest

THE DEVELOPMENT OF MATHEMATICS. *New second edition*

By E. T. Bell, California Institute of Technology. Ready in October

In this revision of a well-known and widely-used book, the author tells with freshness and vigor the absorbing story of the part played by mathematics in the evolution of civilization, from about 4000 B.C. to the present day. The book clearly describes the chief principles, methods, and theories of mathematics which have survived, and presents the greater trends in pure and applied mathematics through major episodes in each period, with explanations of the technicalities. The new edition contains a wealth of new material, much of which is concerned with recent trends and developments in modern mathematics.

INTERMEDIATE ALGEBRA

By R. Orin Cornett, Harvard University, on leave from Oklahoma Baptist University. Ready in October

This is an entirely new approach to the subject. It presents algebra as making possible straightforward, as opposed to trial-and-error solution, of practical problems which would otherwise be difficult or laborious to solve. The nature, advantages, and applications of the algebraic method are made clear before the study of algebraic operations are begun. The treatment of each topic is arranged so as to provide motivation, automatic recognition of significant principles, and mastery of operational techniques.

COLLEGE ALGEBRA. *New second edition*

By Paul K. Rees, Southwestern Louisiana University, and Fred W. Sparks, Texas Technological College. 385 pages, $2.50

Here is a new edition of this popular textbook, presenting a lucid, logical, and teachable treatment of college algebra. In the revision the order of topics has been changed to afford more adequate review material for the student with only one year of high school algebra. Many new problems have been added, in the selection of which particular care has been taken to provide enough variety to emphasize all principles and to furnish sufficient drill.

ANALYTIC GEOMETRY

By Ross R. Middlemiss, Washington University. 308 pages, $2.75

The author of this new text has been influenced in his choice of subject matter and methods by the fact that most students study analytic geometry as a preparation for calculus and the various sciences. Thus an effort has been made to contribute as much as possible to the general mathematical training of the student and to give him a clear understanding of the fundamental methods and most basic concepts of analytic geometry.

Send for copies on approval

McGRAW-HILL BOOK COMPANY, Inc.

330 West 42nd Street **New York 18, N.Y.**

Applied Mathematics

Basic Mathematics for War & Industry

By DAUS, GLEASON, & WHYBURN

Clearly explains all mathematical principles and procedures needed by machinists, carpenters, radio men, and others in industry, the skilled trades and the armed forces, with many illustrative examples and 800 excellent problems for training. Contains especially fine sections on geometry and trigonometry. $2.00

Practical Analytic Geometry with Applications to Aircraft

By ROY A. LIMING

This book makes generally available for the first time the new and highly efficient techniques developed at North American Aviation, Inc., for the design, lofting, and tooling of streamline bodies. $4.50

Navigational Trigonometry

By RIDER & HUTCHINSON

A thorough training in the mathematics basic to navigation, with extensive application to modern sea and air navigation, including the sailings, charts and lines of position, is given in this book. Nomenclature and forms of calculation are those used in actual practice in the armed forces. All problems are realistic. $2.00

Aircraft Mathematics

By WALLING & HILL

This book clearly and simply explains the basic principles and procedures of arithmetic, algebra, graphs, geometry, logarithms, and trigonometry. Hundreds of practical problems give thorough training in the use of these basic procedures for the many kinds of calculations constantly required of those in the air forces. *New American Edition*, $1.25

An Introduction to Navigation and Nautical Astronomy

By SHUTE, SHIRK, PORTER & HEMENWAY

An excellent text either for individual self-instruction or as a class text, this book insures the thorough training in navigation demanded by the Navy today for service in both sea and air forces. Complete in itself, it requires no supplementary books, tables, charts, etc. Conforms throughout to official naval standards and contains over 400 practical problems. $4.50

The Macmillan Company, 60 Fifth Ave., New York 11

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN, U.S.A.

VISUAL AUDIO TAPE

UM

8

CHNOLOGY DEPT.

OCT 31 1945

THE AMERICAN MATHEMATICAL MONTHLY

DEVOTED TO THE INTERESTS OF
COLLEGIATE MATHEMATICS

VOLUME 52 NUMBER 8

CONTENTS

What is the Laplace Transform? . . . . . D. V. WIDDER 419
Circle-to-Line Transformations . . . . . JOHN DE CICCO 425
An Electromagnetic Analogy in Mechanics . . . . . A. WEINSTEIN AND J. R. POUNDER 434
Discussions and Notes . . . . . IRVING KAPLANSKY, W. J. STERNBERG, H. D. GROSSMAN AND DAVID KRAMER 439
Clubs and Allied Activities 443
Recent Publications 448
Problems and Solutions 457
News and Notices 468
General Information 470
- Mathematics in Engineering Graduate Study 470
- Predoctoral Fellowships in the Natural Sciences 471
- New NROTC Units 472
- Postwar Educational Services for Veterans and Service Personnel . 472

The Mathematical Association of America 475
- Wisconsin Section Meeting 475
- Maryland, District of Columbia, Virginia Section Meeting . . 477
- Minnesota Section Meeting 478
- Calendar of Future Meetings 480

OCTOBER 1945